A game theoretic model and a coevolutionary solution procedure to determine the terminal handling charges for container terminals

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ABSTRACT

A game theoretic model was proposed for multiple container terminals competing with each other to maximize their own profits by determining their terminal handling charges (THCs) that affect the market shares of terminals. The pricing model can describe both the competitive and cooperative games. The congestion cost of the terminals was also considered in the cost term, and revenue sharing rental fee schemes with two-step unit fees were analyzed. To overcome the difficulty in obtaining the Nash Equilibrium of the THCs for container terminals, this study proposed a coevolution-based procedure that adopts the neighborhood structure on toroidal grids and supports localized interactions among species. The numerical experiments showed that both the cost model with the congestion cost and the revenue sharing scheme help improve the total profit of the port. In addition, the results obtained by the coevolution-based procedure in this study were compared with those in previous studies.

1. Introduction

With the rapid growth of international trade and economic globalization, the global container cargo volume has been increasing rapidly. The percentage of container cargo volume has increased drastically from 2.75% in 1980 to 6.14% in 1990, 10.49% in 2000, and 16.02% in 2010 (Tsai & Huang, 2017). The global container cargo volumes grew 6.7% in the first half of 2017 (Hand, 2017). In response to the increasing demand for container cargo volume, some countries have started to construct new ports to increase their competitive power, including inter-competition power and intra-competition power, e.g., South Korea opened Busan New Port in 2010.

On the other hand, constructing new terminals incurs severe competition among neighboring container terminals and results in a decrease in handling charge to a level threatening the profitability of container terminals. Table 1 shows that the terminal handling charge (THC) in Korea decreased by 21.5% on average after the operation of Busan New Port began. As a result, some container terminals are suffering from low profitability, even financial deficit, and the mass dismissal of employees. All the containers terminals are paying fixed rental fees to the Busan Port Authority. The government of Busan Metropolitan City and Busan Port Authority attempted to improve the profitability of the container terminals in Busan by weakening the competition among the terminal operators. This issue was well discussed by Ha, Choi, and Kim (2013), who proposed a revenue sharing rental fee scheme, in which the unit rental fee increases with increasing throughput of a terminal, expecting the increased unit rental fee will discourage the terminals from increasing the market share by decreasing its THC.

One of the elements of Industry 4.0 is the horizontal integration among various processes and among players in the supply chain (i-SGOOP, 2020). The importance of the cooperation among players in the global supply chain was also emphasized by Jamrus, Wang, and Chien (2020). Container terminals are crucial players in the global supply chain. This study investigates how container terminal operators determine the terminal handling charges when they compete with each other and what advantages they can obtain when they cooperate with each other. This study also analyzes how the port authority can attract terminal operators to cooperative decisions by using an appropriate rental fee scheme.

This study proposes a game-theoretic model which describes the competition among terminal operators for maximizing their own profit by optimally determining the terminal handling charge. The model is more general than previous studies in that it generalizes mathematical expressions of the terminal operation cost and the revenue sharing scheme between the port authority and a terminal. A new solution method, called coevolution-based procedure, to solve the generalized model is introduced and applied to the case of Busan port.
The remainder of this paper is organized as follows. Section 2 provides a literature review. Section 3 proposes a new model for the game theoretic THC pricing problem. Section 4 introduces a coevolution-based procedure for obtaining the Nash Equilibrium (NE) in competitive games. Section 5 selects the best coevolution strategy to be used in the coevolution-based procedure and evaluates the performance of the coevolution-based procedure. The results show that the coevolution-based procedure in this study outperforms the approach in previous studies. A case of the Busan port was analyzed using the coevolution-based procedure in Section 6. The conclusions and future studies were discussed in Section 7.

2. Literature review

In the field of genetic algorithm (GA), there have been studies on parallel genetic algorithms (PGAs) which perform several concurrent searches of the solution space (Crainic & Toulouse, 1998). There are two types of parallel genetic algorithms: coarse-grained PGA and fine-grained PGA. A coarse-grained PGA replicates the approach in the standard GA on several subpopulations. A migration operator may be added to exchange information among subpopulations regularly. The fine-grained PGA divides the population into a large number of subsets and each subset is connected to several others in its neighborhood. Each subset may contain a single chromosome. Genetic operators are applied for exchanging information between individuals in the same neighborhood. A typical example of the neighborhood may be defined by the set of neighboring entries when chromosomes are located at a two-dimensional matrix, which is adopted in this study.

Collin and Jefferson (1991) compared two selection/crossover schemes: panmictic scheme in which the selection of chromosome for GA operation is done at the level of the entire population; local scheme in which the selection and crossover are performed locally. It was found that the local scheme finds the optimal solution and resistant to premature convergence because different solutions may be explored at different neighborhoods. By the numerical experiments, it was found that the fine-grained PGA has advantages, compared with the panmictic scheme, including the capability of maintaining the diversity of the solutions and the short computation time for obtaining the solution.

Kohlmorgen, Schmeck, and Haase (1999) compared two PGAs: island model, which is another name of the coarse-grained PGA, and the neighborhood model, which corresponds to the fine-grained PGA. It was shown that PGAs provide better solutions in shorter computational time. It was also found that the island model converges much earlier than the neighborhood model but the latter provides better solutions than the former.

Kauffman and Johnsen (1991) studied the coevolutionary dynamics among different species and provided some examples of the problems to which coevolution approaches may be applied. Price (1997) suggested that a coevolutionary programming approach may be used for solving various game problems. It was shown that the coevolutionary genetic algorithm may be applied to several standard industrial organization games, such as the Bertrand and Cournot competition games with linear demand functions for which the optimal solutions are known. A simple coevolutionary genetic algorithm was used to solve these simple problems, in which a chromosome was selected randomly from the entire population of each player for the fitness evaluation. Rieckmann (2001) attempted to explore the relationship between the theory of genetic algorithm and the evolutionary game theory. Rieckmann (2001) attempted to explain that economic learning via genetic algorithms as a special form of an evolutionary game, which GA learning results in a series of near Nash equilibria approaching an evolutionary stable state. Cau and Anderson (2002) proposed a coevolutionary approach, based on genetic algorithm, for describing competitive electricity market. Trading agents were described to co-evolve their own populations of bidding strategies using a genetic algorithm. However, the evolving strategies, including the fitness evaluation, were similar to those the traditional genetic algorithm. The strategy of fitness evaluation, “all against the best,” was used in which a chromosome of a player is evaluated from the competition between it and the best chromosomes of the other players’ populations. Ficici, Melnik, and Pollack (2000) tested various selection methods in the coevolutionary algorithm for an evolutionary game. Sefrioui and Perlaux (2000) also tested a Nash GA to find Nash Equilibrium for various problems. They found that the Nash GA is faster and more robust in finding the NE in terms of computational time. They compared the Nash GA with the traditional Pareto GA, which has been designed to derive the Pareto optimal solutions that must be different from the NE. They used the best chromosome from every other population as the partners for the evaluation of a chromosome in a population.

Son and Baldick (2004) found that the conventional NE search algorithms might trap into a local NE when a game problem has local NE. To overcome this drawback, they proposed a hybrid coevolutionary programming approach in which a population was maintained for each player and the fitness values of chromosomes were evaluated by randomly selecting a chromosome from each population and the result of the tournament decides the fitness values of the selected chromosomes. They proposed the second version of the coevolutionary programming in which for evaluating the fitness value of a chromosome from a population, the best chromosome from each of the other populations are matched and for fine tuning, after matching, the chromosome is improved for increasing its fitness value. They showed that the hybrid coevolutionary approaches find the NE better than the conventional NE search algorithms, and that the second version of the coevolutionary approach performs better than the first version in finding the NE. Maher and Poon (1996) solved computer-based design exploration problems by using a coevolutionary approach. Kim, Kim, and Kim (2000) solved a line balancing and sequencing problem in mixed model assembly lines by using a coevolutionary approach in which chromosomes from different populations evolve through a localized interaction within a neighborhood as in the case of a fine-grained PGA. Kim, Park, and Ko (2003) solved an integrated problem of process planning and job shop scheduling problem and showed that the integrated problem can be solved efficiently by applying the coevolutionary GA with two populations, one for the process planning problem and the other for the shop scheduling problem. They also used the neighborhood structure for the localized evolution, which is also adopted in this study.

This study proposes a new coevolutionary GA for solving competitive game problems which adopts the neighborhood structure in the population for each player, which has been used in PGAs for maintaining the diversity of the solutions in a population but has never been used for solving competitive game problems. The proposed algorithm will be applied to solve the competitive pricing problem among container terminals.

Several studies have reported the port pricing problems that are related to the problem in this study. Song and Panayides (2002) applied a cooperative game theory to the analysis of the cooperation among members of strategic shipping liner alliances. Meersman, Van de
Voorde, and Vaneelslander (2003) introduced a range of factors influencing the port service price. De Borger, Proost, and Van Dender (2008) developed a two-stage game in capacities and prices to analyze the interactions between the pricing behavior of the ports and the optimal investment policies in the port and hinterland capacity. Ishii, Lee, Tezuka, and Chang (2013) developed a non-cooperative game model to analyze the effects of inter-port competition by selecting port charges strategically in the timing of port capacity investment. Park, Min, and Sung (2015) proposed a mathematical model based on cooperative game theory to help port authorities determine the optimal berthing charge, which is a critical element of port pricing. Veldman, Garcia-Alonso, and Vallejo-Pinto (2013) used a logit model to allocate demand among multiple transportation routes, including ports. Woo, Pettit, Kwak, and Beresford (2011) introduced the logit model, which has been popular for modeling the allocation of transportation demand into multiple routes for freight and passengers in transport economics in their survey paper.

Studies specifically related to the pricing problem of the handling charge in container terminals, which is the main issue of this study, have been reported. Chen and Liu (2014) examined the optimal concession contracts considering the fixed-fee, unit-fee, and two-part tariff contract schemes offered by a landlord port authority considering competition among the operators of container terminals using a two-stage game. Chen and Liu (2015) studied the optimal concession contracts offered by a landlord port authority, who pursued traffic-volume maximization, to competing operators of container terminals. Chen et al. (2017) extended the study by Chen and Liu (2014) under the assumption that terminal operators are competing in prices to maximizing their own profit, instead of competing in quantities. Based on the Cournot model developed by Chen and Liu (2014), Liu, Chen, Han, and Lin (2018) analyzed the optimal concession contracts of a port authority with the minimum throughput constraint and with the objective of maximizing the profit of the terminal operators. Han, Chen, and Liu (2018) examined the optimal concession contracts offered by landlord port authorities with two different goals: maximizing the weighted sum of fee revenues and throughput benefits, and maximizing social welfare. The above studies (Chen & Liu, 2014, 2015; Chen et al., 2017) have limitations in that they used a linear function to represent the relationship between the THC and the amount of containers handled by a container terminal operator, which this study attempts to overcome.

Saeed and Larsen (2010a) used a nonlinear function to represent the relationship among the THCs of container terminals and the amounts of containers handled by container terminals. A logit model was used to allocate the total demand to multiple competing container terminals. Although the cooperative game was addressed, the Bertrand Nash Equilibrium was the key element for the analysis. The analytic expression for Bertrand Nash Equilibrium (NE) is derived, which will be compared with the results from this study in section 6. Saeed and Larsen (2010b) analyzed the effects of different types of concession contracts on the revenue of the port authority, the profits of the container terminals, and the port user surplus. Their results are again based on Nash Equilibrium analysis. Saeed and Larsen (2013) provided detail derivations of the analytic expressions for the Nash Equilibrium. Park and Suh (2015) proposed non-cooperative and cooperative game models based on the study by Saeed and Larsen (2010a) considering the case of Busan port. Recently, Zhou & Kim, 2019 studied a two-stage game for designing concession contract between a port authority and container-terminal operators by revenue-sharing schemes with quantity discount.

Compared with previous studies, the contributions of this study are summarized as follows. (1) This study generalizes the operation cost function of a terminal, which was assumed to be proportional to the throughput in previous studies, to be dependent on the capacity utilization of the terminal. (2) This study introduces a new revenue sharing scheme for the rental fee of a terminal, which has a two-step unit rental fee: the first unit rental fee for the throughput below a threshold break point and the second unit rental fee for the throughput above the break point. Note that the two-step unit rental fee is a generalized version of the single unit rental fee in previous studies. (3) To overcome the difficulties in obtaining the Nash Equilibrium using mathematical analysis, this study suggests a coevolution-based method, which is expected to handle more complicated demand allocation functions or profit functions. (4) The proposed model and the coevolution-based procedure are applied to the case of Busan port. The numerical experiments showed that both the proposed operation cost function, considering the capacity utilization and the new revenue sharing scheme with a two-step unit fee, help to improve the total profit of the port. (5) A numerical experiment compared the solution approach in this study with the one in Saeed and Larsen (2010a) and showed that the procedure in this study obtained more accurate Nash equilibria (NEs) than Saeed and Larsen (2010a).

Regarding the third contribution above, this study applied a coevolutionary algorithm to the problem of determining the THC of container terminals competing in the same market for the first time. Most previous studies on this problem (Chen & Liu, 2014, 2015; Chen et al., 2017; Han et al., 2018; Liu et al., 2018) have used analytic methods which have limitations on the complexity of the models that can be solved. Although Saeed and Larsen (2010a, 2010b, 2013) solved a nonlinear model analytically, it will be shown, at the end of Section 5, that the NE solution obtained in Saeed and Larsen (2010a, 2010b, 2013) was not as accurate as the algorithm in this study can obtain. This study assumes a more complicated relationship, the logit model, between the THC of a container and the amount of containers handled by the container terminal operators. The pricing model based on the non-linear functions may not be solved by analytic approaches and the closed form analysis may not be possible. This study attempts to overcome the difficulty through the use of a coevolution based procedure. That is, the necessary conditions for the NE cannot be derived in an explicit form because the profit function is an implicit function and the relationship between the THC of a container and the amount of containers handled by each container terminal operator is also expressed as an implicit function. Thus, this study proposes a new coevolution-based procedure to obtain the NEs in competitive games. The coevolutionary GA, which is a part of the coevolution-based procedure in this study, adopts the neighborhood structure on toroidal grids, which has been used for the evolution process in parallel GAs and which aims at maintaining the diversity of the solutions in a population but has never been used for solving competitive game problems. The coevolution-based procedure proposed in this study is designed for finding NE solution efficiently, because search strategies are selected through extensive numerical experiments. It is expected that the coevolution-based procedure in this study can be applied easily to a wide range of competitive game problems with minor modifications.

3. A competitive pricing problem for container terminals

Container terminals in the same port compete with each other to maximize their profit by attracting more cargo. Although there are many criteria for shipping lines to select a terminal to call in a port, such as handling facilities, service level, turnaround time of vessels and road trucks, storage space, and free time limit for storage, the THC per container is one of the most important criteria. A lower THC will attract more cargo considering the competition with other terminals. On the other hand, this problem must be described as a game because one terminal has to consider the reactions of the other terminals when determining its THC.

This section describes the competitive pricing problem for container terminals. The competition among container terminals is described as a Bertrand game, in which the price is the decision variable of participating players for the competition. Before introducing the behaviors of terminal operators, the notations used in the formulation are listed,
which are almost the same as those used by Saeed and Larsen (2010a):

Indices:
i, j: Indices for a container terminal, i, j ∈ [1, 2, 3, ...].

Parameters:
$CO_i$: Fixed cost component of a terminal user i, who represents an agent who pays the cost of container cargo and determines which terminal to use for their cargo.
$CAP_i$: Annual handling capacity of terminal i, which is determined by the number of berths, the number of quay cranes, areas of the storage yard, and number of yard cranes and yard trucks.
$a_i$: Alternative specific constant of the utility function of terminal i, which represents the attribute of terminal i, enabling it to obtain a high market share compared to other terminals.
w_i: Rental fee paid by terminal i per TEU (handled by terminal i) to the port authority.
$\delta$: Rental fee (in the percentage of the handling price) per TEU paid by terminal i to the port authority.
r_i: Annual rental fee paid by terminal i to the port authority. Note that this rental fee is not dependent on the number of containers handled by terminal i.
$o_i$: Operation cost (per minute) of a gang, which includes the labor cost and equipment-related cost of a quay crane, yard crane.
y: Factor that converts the number of containers to TEUs (Twenty-foot-Equivalent Units) (15 < y ≤ 2).
b_i: Coefficient of the price in the utility function of container terminals.
$A_i$: Base value of the aggregate handling demand function.
$\theta$: Parameter for the aggregate handling demand function $\theta \in [0, 1]$.

Sets:
$T$: Set of all the container terminals.
$T_k$: Set of container terminals in coalition $k$.

Decision variables:
r_i: THC per TEU at container terminal i.

Derived variables:
$x_i$: Yearly container volume handled by container terminal i.
$OUC_i$: Other user costs for container terminal i.
$U_i$: Market share of container terminal i.
$V_i$: Utility function for container terminal i.
$X$: Total yearly aggregate demand.

Cost functions:
$\beta \left( \frac{x_i}{CAP_i} \right)$: Waiting cost function of shippers at container terminal i whose throughput and capacity is $x_i$ and $CAP_i$, respectively.
$\delta \left( \frac{x_i}{CAP_i} \right)$: Operation cost function of container terminal i whose throughput and capacity is $x_i$ and $CAP_i$, respectively. When this is assumed to be constant, this will be denoted as $c_i$.

3.1. Allocating demand for container handling services to terminals

A port terminal levies THCs on terminal users for the handling and storing of containers. This study basically considers the effects of THC on the amount of cargo handled by each terminal. On the other hand, the decision regarding the choice of terminal is affected not only by the THC, but also by some additional cost which is called “other user costs” (OUC). OUC consist of (1) hinterland transport costs (i.e. rail and truck transport costs for moving containers between container terminals and the source locations of the cargo); (2) freight rates charged by carrier lines, including any surcharges related to the port and terminal efficiency; and (3) costs related to the transport time. The cost terms for OUC may be classified into terms related to the operation efficiency of a terminal and those with no relationship to the operation efficiency. The operation is slowed down as the throughput approaches the capacity of a terminal. Woo, Song, and Kim (2016) showed that the operation cycle time of quay cranes becomes longer as the utilization of storage space becomes higher using empirical data.

$OUC_i$ can be expressed generally as follows (Saeed & Larsen, 2010a):

$$OUC_i = CO_i + \beta \left( \frac{x_i}{CAP_i} \right)$$

where $\beta$ is an increasing function of the ratio, $\frac{x_i}{CAP_i}$, and $CO_i$ is a constant value. The utility function of terminal user i can be expressed in the following form:

$$U_i = a_i + b(p_i + OUC_i)$$

where $a_i$ is the alternative specific constant for terminal i and $b$ is the coefficient of the THC and OUC at the terminals.

The logit model has been adopted widely to allocate demand among multiple transportation routes, including ports (Bovy & Bliemer, 2006; Papacostas & Prevedouros, 2001; Veldman et al., 2013). This study assumed that the market share of terminal i can be expressed as Eq. (3) using the logit model,

$$Q_i = \frac{e^{U_i}}{\sum_j e^{U_j}}$$

The total aggregate demand (TEUs) for all the terminals is given by

$$X = Ae^{\alpha s}$$

where $A$ and $\theta$ are constant and $0 < \theta < 1$ (Saeed and Larsen, 2010a, 2010b) and

$$LS = \ln \left( \sum_j e^{U_j} \right)$$

The total aggregate demand may be allocated to each terminal using

$$x_i = XQ_i$$

For given values of THCs of terminals and the constant parameters, the values of $x_i$’s satisfying equations (1)-(6) can be obtained using the Newton-Raphson method (Ypma, 1995). Details of the Newton-Raphson method in this study are described in Appendix A.

3.2. Various profit functions for container terminals

The following three methods for charging rental fees may be assumed. The first rental fee model is a fixed rental fee, which is the one rental system currently used in Korea. The profit function of container terminal i becomes

$$\Pi_i = \left( b_i - \delta \left( \frac{x_i}{CAP_i} \right) \right)X_i - r_i$$

The model proposed by Saeed and Larsen (2010a) and Park and Suh (2015) is a special case of this model with $\delta \left( \frac{x_i}{CAP_i} \right) = c_i$.

The second rental fee model is based on a fixed fee per TEU handled at a container terminal. Let the rental fee be a function of $x_i$, $v_1(x_i) + \tau_i$, which is a revenue sharing model. The profit function of container terminal i can be expressed as

$$\Pi_i = \left( b_i - \delta \left( \frac{x_i}{CAP_i} \right) \right)X_i - v_1(x_i) - \tau_i$$

The model proposed by Saeed and Larsen (2010a) is a special case of this model, where $v_1(x_i) = w_i x_i$. The third type of rental charge is a function of the revenue of the terminal, $v_2(x_i)$. The profit function of container terminal i can then be expressed as

$$\Pi_i = \left( b_i - \delta \left( \frac{x_i}{CAP_i} \right) \right)X_i - v_2(x_i) - \tau_i$$

In the study by Saeed and Larsen (2010a), a special case of this model was proposed: $v_2(x_i) = \delta p_i x_i$. The assumption by Saeed and Larsen (2010a, 2010b) that $\delta \left( \frac{x_i}{CAP_i} \right)$ is a constant, $c_i$, means that the operation time per container remains constant, even when the throughput approaches the handling capacity of the terminal, which is not realistic. In addition, they considered only the variable rental fee models, which is proportional to the total throughput or the total revenue, i.e., $v_2(x_i) = w_i x_i$ or $v_2(p_i x_i) = \delta p_i x_i$. This study will explore the impact of the variable revenue function with a two-step unit fee on the total profit of the port.

To estimate the operation cost for different levels of capacity
utilization of container terminals, Woo et al. (2016) conducted statistical analysis of real data collected from a terminal in Busan and proposed the following relationship: (average the quay crane cycle time per move) = c1 (capacity utilization) + c2. Although they used “storage capacity utilization,” under the assumption that handling capacity and storage capacity are well balanced, the same relationship will be used in this study. That is, this study assumed that the operation cost increases linearly as the capacity utilization increases, which may be expressed by

\[ g_i \left( \frac{x_i}{CAP} \right) = a_1 \left( \frac{x_i}{CAP} + c_2 \right) / \gamma. \]  

(10)

For any contract between the port authority and terminal operators, the profit of any terminal operators should be non-negative, that is, \( \Pi_i \geq 0 \).

In conclusion, the problem in this study can be defined as

\[ \max_{\Pi_i} \Pi_i \text{ for all } i \in T \]  

subject to

\[ x_i = A \exp \left[ \beta \exp \left( \sum_j \exp(U_j) \right) \right] \cdot \left( \exp(U_i) / \sum_j \exp(U_j) \right) \]  

(12)

\[ U_i = a_i + b \left( \frac{x_i}{CAP} \right) \]  

(13)

\[ \Pi_i \geq 0 \]  

(14)

Eq. (12) comes from Eqs. (3), (4), (5), and (6) and Eq. (13) may be obtained by combining Eqs. (1) and (2). The next section proposes a coevolution-based procedure to solve problems (11) - (14) for the following reasons. Even for the simplified case that \( g_i \left( \frac{x_i}{CAP} \right) = c_1 \), \( v_1(x) = w_i x_i \), and \( v_2(p_i) = \delta p_i x_i \), which was assumed in Saeed and Larsen (2010a, 2010b, 2013), from the necessary conditions, \( \frac{\partial^2 \Pi_i}{\partial x_i^2} = 0 \) for all \( i \), it is impossible to derive the explicit equations that can be solved by a numerical search (See Appendix C). The solution obtained by Saeed and Larsen (2010a, 2010b, 2013) involves finding the Nash Equilibrium, maximizing (11), directly need to be developed.

The procedure of the proposed coevolutionary genetic algorithm can be summarized as follows:

Step 1. Initialize the population for each player. The initialization can be done by randomly selecting solutions.

Step 2. Randomly select a player, \( h \). Select an arbitrary chromosome randomly from the population for player \( h \).

Step 3. Identify the scope of the evolving neighborhood (EN) of the selected chromosome. Select two parents from EN using the roulette-wheel selection method (Lipowski & Lipowska, 2012).

Step 4. Generate two offspring by a crossover operation. Replace two individuals with the worst fitness value in EN with the two new offspring and then perform the mutation operation.

Step 5. Update the fitness values of the two offspring using the fitness evaluation procedure, which is described below.

Step 6. If any termination condition is satisfied, then stop; otherwise, go to step 2. The two termination conditions are as follows: (1) the coevolution process runs for a predefined number of generations; and (2) the deviation of the chromosome for each species is smaller than the pre-specified threshold, \( \psi \).

The CoGA in this study is similar to the traditional genetic algorithm but different in that (1) the latter has a single population, whereas the former has a single population for each player; (2) the latter evaluates the fitness value of a chromosome using the information included only in the corresponding chromosome, whereas the former evaluates the fitness value of each chromosome by matching the chromosome with each partner chromosome from the population of each player.

In CoGA, the fitness value of a chromosome may be evaluated by matching one partner chromosome from each of other players. The following introduces various strategies for the selection of partners.

4.1. Fitness evaluation strategies for CoGA

Because there are multiple populations competing with each other and the profit of each player can be calculated only if a partner chromosome of each of other players is given, the main issue for CoGA is how to select the partners to use for evaluating the fitness function, for
which various alternatives are proposed in the following.

(1) Scope of partners

- Entire population (SE) is the scope of the evolving neighborhood; all the chromosomes in the population of each of the other players are candidate partners for the fitness evaluation.
- Toroidal grids (ST) are the scope of an evolving neighborhood. Instead of considering all the chromosomes in the population, only chromosomes, whose positions are in the neighboring toroidal girds of the chromosome to be evaluated, are included in the evolving neighborhood. Chromosomes in the same positions in the population of the other players, as the evolving neighborhood are candidate partners for the fitness evaluation. This will be explained in more detail later.

(2) Partner selection rules

- The best partner in the scope (PB) - the chromosome with the highest fitness value in the scope of partners is selected as a partner for the fitness evaluation.
- Single partner by fitness-based random selection (PF) - a chromosome in the scope of partners is selected randomly based on the fitness values of the chromosomes as a partner of each player.
- Multiple partners by fitness-based random selection (PM) - a pre-specified number of partners are selected randomly from the scope based on their fitness values. All the selected partners of the other players are matched with the chromosome under consideration and the average of all the fitness values of all the combinations of chromosomes is used as the fitness value of the corresponding chromosome.
- All the partners in the scope (PA) - all the chromosomes in the scope of the other players are matched with the chromosome under consideration and the average of all the fitness values of all the combinations of chromosomes is used as the fitness value of the corresponding chromosome.

In the following, the neighborhood scope of toroidal grids (ST) and localized interactions among species are explained in more detail (Kim et al., 2000, 2003). The individuals from a given player are mapped into toroidal grids. Let individual \((m, n)\) denote an arbitrary location on the toroidal grids of the chromosome, for which a neighborhood is defined for the evolution process, and \(N_{\text{sa}}\) denotes \(3 \times 3\) neighbors of the individual \((m, n)\).

For the evolution of populations, first, one species is selected randomly, which called the evolving species. The other species are called symbiotic species. Next, a location \((m, n)\) is selected randomly and the neighborhoods of the evolving species and the symbiotic species are specified. The neighborhood of the evolving species is denoted as \(EN = N_{\text{sa}}\) and the neighborhood of an arbitrary symbiotic species is denoted as \(SN = N_{\text{sa}}\). Two parents are selected from \(EN\) using the Roulette-wheel selection method (Lipowski & Lipowska, 2012) and two offspring are generated through a crossover operation. The worst two individuals in \(EN\) are replaced with the two offspring. For the mutation operation, chromosomes from \(EN\) are selected randomly by using the mutation probability. The mutation operation is implemented on the selected chromosomes. Let the evolving species corresponding to the \(i^{th}\) player and a symbiotic species corresponding to the \(j^{th}\) player.

To evaluate the fitness value of \(P_{\text{a}}\in EN\), which is the individual at \((a, b)\) belonging to the evolving species \(h\), a partner \(P_{\text{a}}\in SN\), which is the individual with the best fitness value belonging to the symbiotic species, for each of the other players is selected. With the set of selected chromosomes, the fitness value of \(P_{\text{a}}\) is evaluated. Information on partners is stored together with the fitness value for each chromosome.

Fig. 2 gives an example of a localized interaction among species. In this example, there are four players. Let species one be the evolving species and the other species be the symbiotic species. Suppose that the chromosome at \((3, 3)\) of species one is selected as the initial point for the evolution and the fitness value of \(P_{\text{a}}\) is to be evaluated. In the following, a range of strategies for partner selection, when evaluating the fitness value, are described. A numerical experiment will be conducted to select the best among the strategies suggested below.

The best partner in scope (PB) - let \(P_{33}, P_{34}, P_{22}\) and \(P_{32}\) be the individuals with the best fitness value belonging to the \(SN\) in each symbiotic species. \(P_{\text{a}}\) is evaluated using \(P_{33}, P_{34}\) and \(P_{22}\) to combine a complete integrated solution.

Single partner by fitness-based random selection (PF) - assume that the maximum number of candidate partners to be selected for the evaluation is set to four. For species two, suppose that \(P_{23}, P_{24}, P_{33}\) and \(P_{34}\) are four chromosomes with the highest fitness values in the \(SN\). One partner among the four candidates is selected randomly with the selection probability dependent on their fitness values. Partners are selected in the same way for the other two species. \(P_{\text{a}}\) is evaluated using the partners selected from each species.

Multiple partners by fitness-based random selection (PM) - assume that the maximum number of candidate partners to be selected for the evaluation is also set to four. Four partners are selected randomly from the scope of each species based on their fitness values. A total of \(4 \times 4 \times 4\) combinations of partners of three species are then made. The average value of the fitness values of species one of 64 integrated solutions becomes the fitness value of \(P_{\text{a}}\).

All the partners in the scope (PA) - all the chromosomes in each \(SN\) become candidate partners. A total of \(9 \times 9 \times 9\) combinations of
partners of three species are then made. The average value of 729 fitness values of species one obtained from 729 integrated solutions becomes the fitness value of $P_{34}$.

### 4.2. Selection of one integrated solution

After applying the CoGA, each player obtains a set of its chromosomes in its population, each of which has a fitness value and partners collected from the other population during the fitness evaluation of the corresponding chromosome. Each player has a set of integrated solutions each of which consists of the chromosomes for all the players. Therefore, it is necessary to derive an agreement on the final solution among all the players. The final agreement should satisfy each player as much as possible. This section proposes a bidding process for deriving agreement among players with multiple alternative solutions from the CoGA. In the bidding process, each player submits one solution in each round until all the players agree with the same solution among all the solutions submitted thus far.

Players may agree with a solution if the solution satisfies the following two conditions:

1. It is one of the Pareto optimal solutions among all the solutions the players have.
2. When the bidding process is applied, no other solution gives a higher profit to each player than the agreed solution.

To satisfy the above two conditions, the dominated solutions are removed from the solutions in the population of each player resulting from CoGA. This step will provide the Pareto optimal solutions for each population. The Pareto optimal solutions remaining at the population of each player are ordered in non-increasing order of payoff to the corresponding player. Before introducing the sequential bidding procedure (SBP), the notations used in the SBP are listed as follows:

**Notations:**
- $T$: set of players
- $U_j$: set of all the Pareto-optimal integrated solutions of player $j$.
- $d$: integrated solution.
- $d_j$: $i^{th}$ Pareto-optimal integrated solution of player $j$.
- $s_j(d)$: solution of player $j$ in an integrated solution $d$.
- payoff($d$): payoff of player $j$ in an integrated solution $d$. This is the profit of player $j$ in a competitive game, while this is the total profit of a coalition in a cooperative game.
- payoff($i,j$): payoff of player $j$ in the $i^{th}$ Pareto-optimal integrated solution of player $j$.

The bidding procedure may be described as follows:

**Step 1:** $i = 0$, $S = \emptyset$.

**Step 2:** $i = i + 1$. If there is no solution in $U_j$ for any player $j$, then conclude that there is no agreed solution and stop. Otherwise, put $d_0$ for all $j \in T$ to the solution set, $S$, and remove $d_0$ from $U_j$ for all $j \in T$.

**Step 3:** Check whether there exists any $d \in S$, such that $\text{payoff}(d_j) \leq \text{payoff}(d)$ for all $j \in T$. If no, then go to Step 2. Otherwise, go to Step 4.

**Step 4:** $d^* = d$. If there multiple $d^*$’s exist, then select $d$ with the largest total payoff, as $d^*$. Stop.

Let $\text{DIS}(d_j)$ represent the level of dissatisfaction of $d_j$ to player $k$, which satisfies the following conditions: (1) if $\text{DIS}(d_j) > \text{DIS}(d_{i-1})$ for all $i$ and $j$; and (2) $\text{DIS}(d_j) = \text{DIS}(d_0)$ for all $i$, $j$, and $k$. The following property holds:

**Property 1:** When SBP is terminated with a single agreed integrated solution, $d$, then the solution $d$ minimizes the maximum dissatisfaction of players.

The proof of property 1 is provided in Appendix B. Table 2 presents an example of the application of SBP. Suppose that the payoff data in Table 2 is obtained by applying CoGA. The integrated solutions for each player are sorted in decreasing order of the profit of the corresponding player. When two players submit their 4th bids, player 1 will accept the 4th bid proposed by player 2 because the profit of player 1 in the 4th bid by player 2 is larger than the profit of player 1 in the 4th bid proposed by player 1. In other words, they arrive at an agreement. The final agreed prices will be $(15, 9)$, which gives the profits $(81, 72)$.

### 4.3. Local search procedure

The selected integrated solution may be close to the NE solution but they require fine-tuning using a local search to make them to arrive at the exact position of the NE. The iterative local search algorithm optimizes the solution of a player to maximize the profit of the player with fixed solutions of the other players until no player can increase its profit or a termination condition is met.

### 4.4. Applying the coevolution-based procedure to the competitive pricing problem

When the coevolution-based procedure in this section is applied to the competitive pricing problem in Section 3, some more detail issues
specific to this problem need to be discussed.

4.4.1. Evaluation of the fitness value

For evaluating the fitness value of a chromosome (THC) for a terminal, the chromosomes (THCs) of the other terminals are selected by using one of the strategies provided in Section 4.1.1. Once the values of $p_i$ are given, $x_i$’s values satisfying equations (12) - (14) can be obtained using the Newton-Raphson method which is explained in Appendix A in detail. The profit of each container terminal operator can be evaluated by equations (7)-(9).

In the simultaneous game with incomplete information, each player makes a decision without knowing the strategy choice of the other players. This study assumes a simultaneous game with incomplete information, which means that each terminal does not know the other player’s payoff (profits). The optimal solution for a simultaneous game is the NE. On the other hand, if more than one terminal forms a coalition, it is allowed for those in the same coalition to “share” its payoff information with the other terminals in the same coalition, and the terminals consider the payoff of the other terminals in the same coalition at the same time, while competing with each other for their own profit. To motivate different species to cooperate with each other during evolution, the fitness function may be defined to include the total profit term in addition to the individual profit as follows.

Suppose that terminal $i$ is included in coalition $k$. The fitness value of the individual $(s, t)$ in $N_{km}$ may be evaluated using the following expression:

$$F_i(s, t) = \prod_{j \in k} (s, t) - \min_{p_i(\theta) \in N_{km}} \prod_{j \in k} (p_i, q_j) + \sum_{j \in k \setminus i} \rho s \prod_{i \in k} (s, t) - \min_{p_i(\theta) \in N_{km}} \prod_{j \in k} (p_i, q_j)$$

(15)

where $\rho \in [0, 1]$ represents the ratio of considering the profit of other terminals in the evaluation of the fitness value and $\prod_{j \in k} (s, t)$ is the profit value of an individual $(s, t)$ for population $i$, which is generated by the interaction. If $\rho = 0$, then each population only considers their own profit, which corresponds to the competitive game. If $\rho = 1$, then the proposed coevolution-based procedure becomes the same as the traditional fully cooperative game. If $0 < \rho < 1$, then the game will be considered to be partially cooperative.

4.4.2. Local search

For searching the best THC with maximizing the profit of a terminal for a given THC of the other terminals, because the decision variable is continuous, the step size of the local search is set to be $\mu$ and the range of the local search to improve each THC per iteration is $[p_i^L - \text{bound}, p_i^L + \text{bound}]$, where $p_i^L$ is the current value. Although $p_i^L$ is a continuous variable, the search space is discretized for the simplicity of the algorithm. In this study, $\mu = 0.0001$ and the bound equaled 5.

4.4.3. Crossover and mutation

As introduced in Section 4.1, from two parents which are selected from $EN$, two offsprings are generated through a crossover operation. The worst two individuals in $EN$ are replaced with the two offsprings. Chromosomes from $EN$ are selected randomly by using a mutation probability. The mutation operation is implemented on the selected chromosomes.

Considering a chromosome is represented by a real value, the crossover operation introduced by Haupt and Haupt (2004) was adopted. A chromosome is represented by a value between 0 and 1. For encoding the terminal handling charge, it was normalized to a value in [0,1] in which the lower and the upper bound corresponds to 0 and 1, respectively.

Let $p_{im}$ and $p_{in}$ are the chromosomes of individual $m$ and $n$ in the population of terminal $i$, respectively. The crossover operation can then be expressed by using Eq. (16).

$$p_{in}^m = \beta p_{in}^m + (1 - \beta)p_{in}^n,$$

(16)

where $\beta$ is a random number in the interval $[0, 1]$. For mutation operation, the random resetting method was adopted. That is, the randomly selected chromosome was replaced with a random value that belongs to $[0, 1]$.

5. Numerical experiments for evaluating the proposed coevolution-based procedure

Three experiments were performed to test the proposed coevolution-based procedure. The first experiment attempted to select the best fitness evaluation strategy among all combinations of two strategies for “the scope of partners” and four partner selection rules. The second experiment showed that how the profits of terminals may be improved by sharing information on their profit functions. The third experiment compared the solutions by the coevolution-based procedure in this study with those in previous studies.

5.1. Exploring the best fitness evaluation strategy for CoGA by numerical experiments

The proposed coevolution-based procedure was implemented using C++ programming language on a desktop PC with an Intel(R) Core (TM) i7-4790 CPU at 4.0 GHz and 32 GB of memory. Various experiments were performed to find good strategies of the CoGA. The profit function, assuming $g_i(x_C) = c_i$, $v_i(x_i) = w_i x_i$, and $v_j[x_j(x_i)] = \delta_j x_j x_i$ was used. This study used the data in Table 3 and it was assumed that $A = 800,000$, $\delta = 0.05$ and $b = -0.05$, which are provided by previous studies (Saeed and Larsen, 2010a, 2010b). In the following experiments, terminals Q and P represent QICT and PICT in the studies by Saeed and Larsen (2010a, 2010b), respectively. The two terminals are assumed to compete with each other, i.e., $\rho = 0$.

The following values of parameters for the CoGA were used for the experiment: population size = 100, crossover probability = 0.8, mutation probability = 0.1, Maximum no. of generations = 200000, lower bound of THC = 0, upper bound of THC = 200, and $\psi = 0.0001$. The lower and upper bounds of the terminal handling charge were set as the search space for the optimal solution and thus should be wide enough to include the optimal solution. The lower bound was set to be 0, while the upper bound was set to be a sufficiently large value, 200 USD, which is roughly three times of the largest handling charge, 69,9438 USD, in Busan New Port in Table 1. The following expression, which was used by Saeed and Larsen (2010b), was assumed to express the other user cost in the numerical experiment:

$$f_i \left( \frac{x_i}{\text{CAP}} \right) = 0.5 * \left( \frac{x_i}{0.8 * \text{CAP}} \right)^4.$$ 

A total enumeration method was used to find the NE for the two terminal game, in which all the values of the THC of a terminal were

<table>
<thead>
<tr>
<th>Table 3</th>
<th>Data for two terminals (Saeed &amp; Larsen, 2010a).</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters</td>
<td>Terminals</td>
</tr>
<tr>
<td>Container terminal name</td>
<td>Q</td>
</tr>
<tr>
<td>Alternative specific constant ($c_0$)</td>
<td>0.1</td>
</tr>
<tr>
<td>User cost constants in $/C_0$</td>
<td>7</td>
</tr>
<tr>
<td>Marginal cost in $/C_0$</td>
<td>50</td>
</tr>
<tr>
<td>Capacity (CAP)</td>
<td>600,000</td>
</tr>
<tr>
<td>Terminal fee in $/C_j$</td>
<td>5% of price</td>
</tr>
<tr>
<td>Terminal fee in $/w$</td>
<td>–</td>
</tr>
<tr>
<td>Annual rental fee in 1000$/C_i$</td>
<td>800</td>
</tr>
</tbody>
</table>
enumerated to maximize its profit for each fixed value of the THC of the other terminal. To obtain the NE, the value of $p_2$, which maximizes the profit of terminal Q for each fixed value of $p_2$, is obtained. Insert the obtained solution, $(p_2, p_0)$, into $A$. This process is repeated for all the possible values of $p_2$. The value of $p_2$, which maximizes the profit of terminal P for each fixed value of $p_2$, can then be obtained. Insert the obtained solution, $(p_2, p_0)$, into $B$. This process is repeated for all the possible values of $p_2$. The pair of $(p_2, p_0)$, which is included in both $A$ and $B$, is selected as the NE.

The effectiveness and efficiency of the various solution strategies for CoGAs were compared with each other. Because all the solution strategies obtained the NE, their calculation times were compared with each other. The calculation time of the coevolution-based procedure consists of two parts: the calculation time for CoGA and that for the iterative local search.

Table 4 summarizes the calculation time of the coevolution-based procedure. Two termination conditions for the coevolution genetic algorithm were used in this study. The first termination condition is the maximum number of coevolving generations. The second termination condition was that the procedure stops when the deviation of fitness values becomes smaller than a pre-specified threshold value for all populations. If one between two holds, then the coevolution genetic algorithm will stop. When the algorithm is stopped by the second termination condition, different strategies may have a different number of iterations. The ST outperformed SE for any partner selection rule in terms of the computational time. The number of partners selected for iterations. The ST outperformed SE for any partner selection rule in terms of the computational time. The number of partners selected for the fitness evaluation from one population, affected the computational time of the coevolution genetic algorithm significantly. Therefore, the CoGA with the entire population as the neighborhood (SE) always requires more computation time than the Toroidal grids (ST). For the same reason, PM and PA strategies, which require a larger number of fitness evaluations, took a longer computational time. In addition, the solution by the random selection rule (PF) for the partners based on the fitness converges to the final solution faster than the rule selecting the partner with the highest fitness value. The strategy, ST + PF, was used for the experiments in the following sections.

Fig. 3 shows the solutions obtained by the coevolutionary learning stage and local search for the case of (ST + PF). The first column in Fig. 3 shows the randomly initialized solutions. The second column shows the solution after 500 iterations. The third column presents the solutions after the coevolutionary learning process is completed. The final column shows the solutions after the iterative local search, which coincide with the NE obtained by the total enumeration.

5.2. Effect of sharing payoff information on the profits of terminals

This section evaluates the effects of various levels of information sharing among the terminals in a coalition. COMP represents the competitive game with $\rho = 0$. When $\rho > 0$, the game was denoted as "COOP." Fig. 4 shows the results of COMP and COOP for various values of parameter $\rho$. Fig. 4 shows the position of the NE for COMP, which is represented by the star on the lower-left corner in Fig. 4 (a)-1, (a)-3, (b)-1, (b)-3, (c)-1, and (c)-3. The other nodes in Fig. 4 are solutions obtained by applying only CoGA for COOP. Note that the position of the NE is far away from the solutions of COOP. The distances from the NE to the solutions of COOP in both the prices and profits become longer as $\rho$ increases. Fig. 4 (a)-2, (a)-4, (b)-2, (b)-4, (c)-2, and (c)-4 shows a close-up view of a part, which was selected by the small rectangle, of Fig. 4 (a)-1, (a)-3, (b)-1, (b)-3, (c)-1, and (c)-3, respectively. In the figures for profit, solid circle nodes represent the Pareto front among the solutions for COOP. In both figures for the price and profit, the solid circle node with a cross inside indicates the final solution of COOP after applying SBP. Note that in COOP, both the profits of two terminals were significantly increased, which implies that all the terminals may become better off by collaboration among terminals in determining the THCs.

Fig. 5 compares the profit between the COMP and COOP. The dashed single dotted and dashed lines denote the profit of terminal Q and terminal P in the NE solution of COMP, respectively. The lines with the circle and rectangle nodes indicate the profit of terminals Q and P in COOP, respectively. The line with the pentagonal nodes denotes the total profit of terminals Q and P by COOP. Fig. 5 shows that, as the value of $\rho$ increases, the total profit increases monotonically.

5.2.1. Cooperative game

Some container terminals may organize a coalition of container terminals, in which all the container terminals cooperate to maximize the total profit of container terminals involved in the coalition. However, the coalition competes with other terminals outside the coalition. By using the coevolution-based procedure in this study, this cooperative game may be analyzed by setting the objective function of players in the same coalition to be the total profit of all the terminals in the coalition ($\rho = 1$).

5.3. Comparison of the coevolution-based procedure with an approach by previous studies

In this section, to evaluate the proposed approach, different problems including the independent game and cooperative games were studied. In independent game, each container terminal is an

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**Table 4**
Calculation time of the coevolution-based procedure with various strategies for CoGA.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>SE + PB</th>
<th>SE + PF</th>
<th>SE + PM</th>
<th>SE + PA</th>
<th>ST + PB</th>
<th>ST + PF</th>
<th>ST + PM</th>
<th>ST + PA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (Second)</td>
<td>135.1</td>
<td>70.4</td>
<td>79.6</td>
<td>35.56</td>
<td>61.56</td>
<td>24.3</td>
<td>42.2</td>
<td>33.8</td>
</tr>
</tbody>
</table>

---

![Fig. 3. Solutions after CoGA and the final solution for the (ST + PF) procedure.](image-url)
independent player. However, in cooperative games, different container terminal ally as a coalition to compete with other container terminals not belonging to their coalition. The solutions were compared with those provided by previous academic studies (Saeed and Larsen, 2010a, 2010b, 2013), using the same input data and expressions for the demand and profit functions, as in previous studies. Saeed and Larsen (2010a, 2010b, 2013) proposed an analytical method for obtaining the NE solution which showed considerable gaps from true values (see Appendix C).

Table 5 shows the input data of the case provided by Saeed and Larsen (2010a), where terminals QICT, KICT, PICT and KPT in Saeed and Larsen (2010a) are represented by terminals Q, K, P, and T. The rental charge method of Eq. (8) is applied to terminal Q, while that of Eq. (7) is applied to terminals K and P. Note that terminals K, P, and T are located in the same port. The port authority of the port owns terminal T and thus it has the revenue of rental fees collected from terminals K and P. The port authority and terminal T are always considered to be an economic entity which is represented by terminal T. That is, the profit of terminal T consists of two parts: (1) the profit obtained by the container terminal operator and (2) the revenue obtained by the port authority. Finally, the profit function of terminal T can be expressed as

\[ \prod_{T} = (p_T - c_T)x_T + \sum_{j \in \{K, P\}} w_j x_j. \]  \tag{17} 

The parameters for the coevolution-based procedure were assumed to be the same as in the previous experiment for two terminals, which was provided in Table 4. It was assumed that \( A = 1,550,000 \), \( \theta = 0.01 \), and \( b = -0.05 \). All the values of the parameters used in this experiment were the same as those used by Saeed and Larsen (2010a).

5.3.1. Experiments for the competitive game among container terminals

In this experiment, it is assumed that all the terminals compete with each other without any cooperation (\( \phi = 0 \)). Table 6 presents the solution obtained by the proposed coevolution-based procedure (ST + PF). The THCs obtained by the coevolution-based procedure were higher than those by Saeed and Larsen (2010a). On the other hand, each terminal makes one or two million US dollars higher profit than that by Saeed and Larsen (2010a).

Figs. 6 and 7 show the curve of the profit function in the neighborhood of the final solution by Saeed and Larsen (2010a) and this study, respectively. The curve in each figure shows the change in the profit function for various THC values of one container terminal while the THCs of the other three container terminals are fixed at their values in the final solution. Fig. 6 (a), (b), (c) and (d) shows that by changing

(a) \( \rho = 0.2 \)

Fig. 4. COMP and COOP with different values of parameter \( \rho \).
the THC for each container terminal, the terminal can gain more profit, which suggests that the final solution provided by Saeed and Larsen (2010a) is not located exactly at a local NE. Fig. 7 shows that no container terminal can gain more profit by changing its THC, which means that the final solution obtained by the proposed coevolution-based procedure is at a local NE.

5.3.2. Experiments for the cooperative game with coalitions

This section applies the coevolution-based procedure to four cases with different coalitions of terminals K, P, and T in the same port as follows, which was studied in Saeed and Larsen (2010a):

Case A: Cooperation between terminals T and K
Case B: Cooperation between terminals K and P
Case C: Cooperation between terminals T and P
Case D: Cooperation between all three container terminals K, P, and T.

When more than one container terminal are included in the same coalition, it is assumed that \( \rho = 1 \), that is, they become to have the same objective function which is the total profit of all the terminals in the coalition.

Table 7 compares the final THCs from Saeed and Larsen (2010a) and those from the coevolution-based procedure in this study. Table 7 shows that the results from the two solution methods have big gaps. The difference between two solutions ranges between 2.64% and 14.47%.

Let \( p_i \) denote the final THC for container terminal \( i \), which is obtained by an algorithm. And let \( p_i^o \) represent the THC of container terminal \( i \) with maximizing the profit of container terminal \( i \) (or a coalition where terminal is involved in a cooperative game) under the condition that values of THCs of the other container terminals are set to be \( p_j(\ j \neq i) \). Thus, the deviation of the final solution from the peak (Dev) can be written as

\[
\text{Dev}_i = \left| \frac{p_i - p_i^o}{p_i^o} \right| \times 100\%.
\]

Table 8 shows that \( \text{Dev}_i \) obtained by the study of Saeed and Larsen (2010a) ranges between 0.19% and 22.19%, which represents how far the final solutions are located from the NEs. Note that the values of Dev of the result from the coevolution-based procedure are zero, which means that the final solutions satisfy the condition for the NE.

6. Application of the coevolution-based procedure to Busan port

Busan port is the largest port in South Korea and it is operated by the Busan Port Authority (BPA). Busan port consists of two main parts:
The North port and New port. In the Busan port, there are container terminals, such as Korea Express Busan Container Terminal (KBCT), Hutchison Busan Container Terminal (HBCT), Dongbu Pusan Container Terminal (DPCT), Gamman Container Terminal (Gamman), Pusan New Port Company (PNC), and Hanjin New Container Terminal (HJNC). This study analyzed the case of Busan port, where six container terminals are competing with each other ($\rho = 0$). Note that PNC is a privately owned terminal and thus does not pay any rental fee to the BPA. The same input data as those in Park and Suh (2015) were used for the experiments, which are listed in Table 9. The values of the model-demand parameters are as follows: $a_i = 30.111$ for all $i$, $b = -0.046$, and $d = 0.01$. The following expression for the waiting cost function for shippers, as in Park and Suh (2015), was used in this section:

$$w(Q) = C_i(Q) + b(Q) + dQ$$

### Table 5

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Terminals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Container terminal name</td>
<td>Q</td>
</tr>
<tr>
<td>Alternative specific constant ($a_i$)</td>
<td>0.1</td>
</tr>
<tr>
<td>User cost constants in $ (CO_i)$</td>
<td>7</td>
</tr>
<tr>
<td>Marginal cost in $ (c_i)$</td>
<td>50</td>
</tr>
<tr>
<td>Capacity (CAP_i)</td>
<td>600,000</td>
</tr>
<tr>
<td>Terminal fee in $ (\xi)$</td>
<td>5% of price</td>
</tr>
<tr>
<td>Terminal fee in $ (\eta_i)$</td>
<td>–</td>
</tr>
<tr>
<td>Annual rent in 1000$</td>
<td>800</td>
</tr>
</tbody>
</table>

Fig. 4. (continued)

Fig. 5. Profit comparison between COMP and COOP.
BPA has been charging a fixed rental fee of Eq. (7) to the terminals except for the PNC. As mentioned before in Table 1, BPA has been seriously suffering from the low profitability of terminals in Busan resulting from severe competition among the terminals. Thus, BPA attempts to find an alternative rental fee scheme that increase the profitability of the container terminals. Revenue-sharing schemes are candidates as alternatives. This study examined and compared various revenue sharing schemes.

The following four rental fee schemes were considered in this study: one fixed rental fee and three revenue sharing schemes:

(Scheme 1) \[ \prod_{i} = \left( p_i - g_i \frac{x_i}{\text{CAP}} \right)x_i - n_i, \]

(Scheme 2) \[ \prod_{i} = \left( p_i - g_i \frac{x_i}{\text{CAP}} \right)x_i - w_ix_i - \xi, \]

(Scheme 3) \[ \prod_{i} = \left( p_i - g_i \frac{x_i}{\text{CAP}} \right)x_i - w_i(x_i - \text{CAP}) - w_{ji}\text{CAP} - \xi \text{ when } x_i < \text{CAP}; \]

\[ \prod_{i} = \left( p_i - g_i \frac{x_i}{\text{CAP}} \right)x_i - w_i(x_i - \text{CAP}) - w_{ji}\text{CAP} - \xi \text{ when } x_i \geq \text{CAP}; \]

\[ \prod_{i} = \prod_{i} + \prod_{i} \text{ with } w_{ji} < w_{ji} (w_{ji} = (1 + \tau) \times w_{ji}). \]

Note that \( \prod_{i} \) represents the profit function of a terminal when a fixed rental fee scheme is used, as in Park and Suh (2015). \( \prod_{i} \) represents the profit function when a single revenue sharing rate is used, as in Saeed and Larsen (2010a). \( \prod_{i} \) includes the revenue sharing with a discounted unit rate for the throughput exceeding its capacity, while \( \prod_{i} \) applies an increased unit rate for the throughput exceeding its capacity. Note that \( \prod_{i} \) and \( \prod_{i} \) are proposed in this study for the first time. When \( \tau = 0 \) and \( w_{ji} = w_i \), \( \prod_{i} \) and \( \prod_{i} \) become \( \prod_{i} \).

Table 6
Comparison between the procedure in this paper with that in Saeed and Larsen (2010a).

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Terminals</th>
<th>Algorithms</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Saeed and Larsen</td>
<td>This study</td>
</tr>
<tr>
<td>Price (US dollar/TEU)</td>
<td>Q</td>
<td>81.6</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>90.1</td>
</tr>
<tr>
<td></td>
<td>P</td>
<td>91.9</td>
</tr>
<tr>
<td></td>
<td>T</td>
<td>53.2</td>
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<tr>
<td>Handling quantity (TEU)</td>
<td>Q</td>
<td>450,600</td>
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<tr>
<td></td>
<td>K</td>
<td>465,620</td>
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<tr>
<td></td>
<td>P</td>
<td>270,360</td>
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<tr>
<td></td>
<td>T</td>
<td>315,420</td>
</tr>
<tr>
<td>Profit (in million US dollar)</td>
<td>Q</td>
<td>12.4</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>13.8</td>
</tr>
<tr>
<td></td>
<td>P</td>
<td>6.6</td>
</tr>
<tr>
<td></td>
<td>T</td>
<td>14.3</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>47.1</td>
</tr>
</tbody>
</table>

\[ f_1 = 0.6192 \left( \frac{x_1}{\text{CAP}} \right)^3 - 7.2968 \left( \frac{x_1}{\text{CAP}} \right)^2 + 25.051 \left( \frac{x_1}{\text{CAP}} \right) - 17.209 \]  

(18)

Fig. 6. Profit changes in the neighbor of the final solution by Saeed and Larsen (2010a).
To evaluate the various rental fee schemes, the total profit function of the port was used. Because the rental fee paid by a terminal to the BPA becomes the revenue of the BPA, the rental fee was excluded from the total profit of the port, which may be expressed as follows:

\[ SW = \sum_{i=1}^{T} \left( p_i - g_i \left( \frac{x_i}{C_{AP}} \right) \right) x_i. \]

The values of other parameters, which were not provided by Park and Suh (2015), were set as follows:

\[ w_i = \frac{r_i}{C_{AP}} \]

and \( o_i = \gamma C \left( \frac{\text{realized throughput}}{C_{AP}} + e_i \right) \) \tag{23}

First, the model with a fixed operation cost per container, \( g_i \left( \frac{x_i}{C_{AP}} \right) = c_i \), and that including a congestion cost, \( g_i \left( \frac{x_i}{C_{AP}} \right) = o_i \left( \frac{x_i}{C_{AP}} + e_i \right) / \gamma \) \( (\gamma = 1.5, \ e_1 = 1.795, \ e_2 = 0.7547 \text{ min}) \)

### Table 7
Comparison of terminal handling price obtained by different schemes.

<table>
<thead>
<tr>
<th></th>
<th>Saeed and Larsen (2010a) ($)</th>
<th>Coevolution-based procedure ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Q</td>
<td>K</td>
</tr>
<tr>
<td>Case A</td>
<td>79.60</td>
<td>91.60</td>
</tr>
<tr>
<td>Case B</td>
<td>80.40</td>
<td>96.70</td>
</tr>
<tr>
<td>Case C</td>
<td>78.50</td>
<td>89.70</td>
</tr>
<tr>
<td>Case D</td>
<td>83.00</td>
<td>102.00</td>
</tr>
</tbody>
</table>

Woo et al., 2016) were compared. The base values of \( c_i \) are listed in Table 8 and the values of \( c_i \) were varied in steps of 10%. The value of \( o_i \) was varied using Eq. (23), as the value of \( c_i \) changes. Table 10 lists the total profit of the port obtained by the model with the operation cost, including the congestion cost and that with a fixed unit operation cost, when rental fee scheme 1 of Eq. (19) is used. Table 10 suggests that the operation cost model including the congestion cost, which is more
Table 9
Input data for terminals in Busan.

<table>
<thead>
<tr>
<th>Terminals</th>
<th>Handling capacity (million TEU) (CAPi)</th>
<th>Current fixed rental charge (million $) (ri)</th>
<th>COi($)</th>
<th>ci($)</th>
<th>Current handling charge per TEU ($) (pi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>KBCT</td>
<td>2</td>
<td>30.6</td>
<td>19</td>
<td>16.51</td>
<td>41.7</td>
</tr>
<tr>
<td>HBCI</td>
<td>1.7</td>
<td>25.7</td>
<td>16</td>
<td>33.55</td>
<td>48.7</td>
</tr>
<tr>
<td>DPCT</td>
<td>0.8</td>
<td>16.6</td>
<td>14</td>
<td>8.23</td>
<td>37.8</td>
</tr>
<tr>
<td>Gaman</td>
<td>1.6</td>
<td>23.4</td>
<td>16</td>
<td>20.57</td>
<td>35.7</td>
</tr>
<tr>
<td>ILJNC</td>
<td>1.6</td>
<td>22.8</td>
<td>17</td>
<td>22.12</td>
<td>39.5</td>
</tr>
<tr>
<td>PNC</td>
<td>2.8</td>
<td>0</td>
<td>16</td>
<td>29.71</td>
<td>40.1</td>
</tr>
</tbody>
</table>

Table 10
Comparison of the total profit (unit in million US $) of the profit model with fixed unit operation cost and that with congestion cost in rental fee scheme 1.

<table>
<thead>
<tr>
<th>Operation cost</th>
<th>0.8ci</th>
<th>0.9ci</th>
<th>ci</th>
<th>1.1ci</th>
<th>1.2ci</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed operation cost</td>
<td>486</td>
<td>494</td>
<td>502</td>
<td>511</td>
<td>519</td>
</tr>
<tr>
<td>Congestion cost</td>
<td>603</td>
<td>621</td>
<td>639</td>
<td>657</td>
<td>674</td>
</tr>
<tr>
<td>Improvement (%)</td>
<td>24.01</td>
<td>25.56</td>
<td>27.15</td>
<td>28.63</td>
<td>29.98</td>
</tr>
</tbody>
</table>

### 6.1. Managerial implications

The above experiments give the following managerial implications:

1. The inclusion of the congestion cost, which increases with increasing throughput, increases the total profit of the port. Therefore, to increase the profitability of terminals in a port, a more realistic operation cost function considering the negative impact of the congestion in terminals should be used to estimate the profit of terminals.

2. The revenue sharing schemes increases the total profit of the port. Therefore, to increase the profitability of the terminals in a port, the revenue sharing scheme instead of the fixed rental fee scheme should be used. In addition, if possible, the revenue sharing scheme with a penalty for a high throughput should be used.

### 7. Conclusion

This study proposed a coevolution-based procedure to find the Nash Equilibrium of the handling charges for multiple container terminals in a port, which compete among individual terminals or among the coalitions of terminals to maximize their own profit.

The relationships among handling prices and handling demands allocated to containers are expressed using multiple non-linear simultaneous equations, which do not allow closed form expressions for the Nash Equilibrium (NE) solution. To overcome this difficulty, this study proposed a coevolution-based procedure consisting of a coevolutionary genetic algorithm for obtaining solutions close to the Nash Equilibrium solution, a bidding procedure for selecting one solution, and a local search procedure to fine tune the selected solution. It was also discussed how the coevolution-based procedure can be applied to the cooperative game.

This study generalized the operation cost function of the container terminals, which can include the congestion cost and unit fixed cost that have been used in previous studies. Four different rental fee schemes by the port authority have been analyzed, including the fixed, revenue sharing with a single unit fee, revenue sharing with a discounted unit fee, and revenue sharing with a penalty unit fee.

A numerical experiment with two terminals showed that a coevolution-based procedure obtains the NE solution successfully. It was found that the neighborhood structure with toroidal grids is effective in defining the scope of an evolving neighborhood for selecting fitness-evaluating partners.

The solutions from the coevolution-based procedure in this study were compared with that by Saeed and Larsen (2010a) for both the competitive game and the cooperative game. It was found that the coevolution-based procedure in this study obtained the NE solutions more accurately than that by Saeed and Larsen (2010a).

A numerical experiment was conducted to evaluate the effects of the cooperation among terminals by sharing the information on their own profit functions among terminals. It was found that solutions from the collaboration increase the profits of every terminal compared with the NE solution obtained in the competitive situation.

Numerical experiments were performed for the case of Busan port to analyze the effects of using a more realistic operation cost function with the congestion cost and the four rental fee schemes on the total profit of Busan port. Both the inclusion of the congestion cost into the operation cost and the revenue sharing rental fee help increase the total profit of the port. The revenue sharing rental fee scheme with a penalty increases the total profit of the port further.

Future studies address the following issues: (1) a method for a port authority to determine the optimal parameters of the revenue sharing schemes considering the expected responses of the terminals; (2) developing new revenue sharing schemes; and (3) applications of the coevolution-based approach in this study to other competitive game problems.
Acknowledgment

This research was supported under the framework of International Cooperation Program managed by National Research Foundation of Korea (NRF-2016K1A3A1A48954044). The authors appreciate Mr. Jae Gwan Kim for supporting the programming of the Newton-Raphson method and Professor Nam-Kyu Park for supporting the case study of Busan port. The authors also acknowledged the China Scholarship Council (CSC NO. 201608260014) for supporting Yanjie Zhou to study in the Department of Industrial Engineering, Pusan National University.

Appendix A: Newton-Raphson method applied to the allocation of handling demand

Let $\varepsilon$ be the threshold for the Newton-Raphson method and $k_{\text{max}}$ be the maximum number of iterations for the Newton-Raphson method. In this study, it was assumed that $\varepsilon = 0.00001$ and $k_{\text{max}} = 1000$ in the experiments. Let $\vec{X}$ be the vector and $\vec{x} = (x_1, \ldots, x_i)$. $\vec{X}$ will be obtained using an iterative procedure. Let $\vec{X}_{(i)}$ be the value of $\vec{X}$ in the $i^{th}$ iteration. Eq. (6) can be rewritten as Eq. (A1).

$$F(\vec{X}) = x_i - XQ_i = 0.$$  

(A1)

It was attempted to find $x_i$’s satisfying (A1). By taking the derivative of Eq. (A1) with respect to $x_i$,

$$\frac{\partial F(\vec{X}_i)}{\partial x_i} = 1 - \frac{\partial (XQ_i)}{\partial x_i} - X \frac{\partial (Q_i)}{\partial x_i} \quad (A2)$$

By taking the derivative of Eqs. (1) and (3) with respect to $x_i$,

$$\frac{\partial (U_i)}{\partial x_i} = \bar{b} \frac{\partial (OUC_i)}{\partial x_i} \quad (A3)$$

$$\frac{\partial (OUC_i)}{\partial x_i} = \frac{\partial f_i}{\partial x_i} = 0.5 \left( \frac{x_i}{0.8 \cdot \text{CAP}_i} \right)^4 \quad (A4)$$

$$\frac{\partial (U_i)}{\partial x_i} = \frac{2b}{0.8 \cdot \text{CAP}_i} \left( \frac{x_i}{0.8 \cdot \text{CAP}_i} \right)^3 \quad (A5)$$

Eq. (A6) is obtained by replacing the intermediate variables in Eq. (A2) as follows:

$$\frac{\partial F(\vec{X}_i)}{\partial x_i} = 1 - \text{AE}(\text{CAP}_i)^2 \sum_{j=1}^{3} e^{\rho_i} \left( \frac{2b}{0.8 \cdot \text{CAP}_i} \left( \frac{x_i}{0.8 \cdot \text{CAP}_i} \right)^3 \sum_{j=1}^{3} e^{\rho_j} \right) - (\text{AE}(\text{CAP}_i)^2) \left( \frac{2b}{0.8 \cdot \text{CAP}_i} \left( \frac{x_i}{0.8 \cdot \text{CAP}_i} \right)^3 \sum_{j=1}^{3} e^{\rho_j} \right)$$

(A6)

By taking the derivative of Eq. (A1) with respect to $x_j$,

$$\frac{\partial F(\vec{X}_j)}{\partial x_j} = - \frac{\partial (XQ_j)}{\partial x_j} - X \frac{\partial (Q_j)}{\partial x_j} \quad (A7)$$

Using Eqs. (A6) and (A7), the Jacobian matrix of $\vec{X}$ can be represented as
At the \( k \)th iteration, \( J(\bar{X}) \) is formed and the following expression is evaluated repeatedly until the values of \( \bar{X} \) converge

\[
\bar{X}_{(k+1)} = \bar{X}_{(k)} - [J(\bar{X}_{(k)})]^{-1}F(\bar{X}_{(k)})
\]

(A9)

The details of the Newton-Raphson method are summarized in the following.

Step 1. Initialize the parameters and set \( k = 0 \), then initialize \( \bar{X}_{(k)} \).

Step 2. Calculate Jacobian matrix \( J(\bar{X}) \) and update \( \bar{X}_{(k+1)} \) using Eq. (A9).

Step 3. If \( k = k_{\text{max}} \), then the procedure stops with no convergence. Otherwise, go to step 4.

Step 4. Update \( F(\bar{X}_{(k)}) \).

Step 5. If \( F(\bar{X}_{(k)}) \leq \epsilon \), then stop the procedure and return \( \bar{X}_{(k)} \). Otherwise, go to step 2.

Note that when \( j \left( \frac{x}{\text{CAP}} \right) = 0.6192 \left( \frac{x}{\text{CAP}} \right)^3 - 7.2968 \left( \frac{x}{\text{CAP}} \right)^3 + 25.051 \left( \frac{x}{\text{CAP}} \right) - 17.209 \), the same procedure may be used to allocate handling demand to terminals using the following Jacobian matrix:

\[
\frac{\partial J(\bar{X})}{\partial x_i} = 1 - Ae^{eLS}b\left( 1.8576 \left( \frac{x_i}{\text{CAP}} \right)^2 + 25.051 \left( \frac{x_i}{\text{CAP}} \right)^2 \right) - Ae^{eLS}\left( 1.8576 \left( \frac{x_i}{\text{CAP}} \right)^2 + 25.051 \left( \frac{x_i}{\text{CAP}} \right)^2 \right) \left( \frac{e^{i+1}(e^{i+1})}{\left( \sum_{j \in S} e^{j} \right)^2} \right)
\]

(A10)

\[
\frac{\partial J(\bar{X})}{\partial x_j} = (1 - \theta)(Ae^{eLS})b\left( 1.8576 \left( \frac{x_j}{\text{CAP}} \right)^2 + 25.051 \left( \frac{x_j}{\text{CAP}} \right)^2 \right) - Ae^{eLS}\left( 1.8576 \left( \frac{x_j}{\text{CAP}} \right)^2 + 25.051 \left( \frac{x_j}{\text{CAP}} \right)^2 \right) \left( \frac{e^{j+1}(e^{j+1})}{\left( \sum_{j \in S} e^{j} \right)^2} \right)
\]

(A11)

Appendix B: Proof of Property 1

Proof of Property 1: Assume that the iteration is finished at the \( t \)th iteration. Let \( S_{(t)} = \{ \text{payoff}_j(d_i) \leq \text{payoff}_j(s) \} \) and \( S \) at the \( t \)th iteration. From the condition that there is a single \( d \) satisfying such that \( \text{payoff}_j(d_i) \leq \text{payoff}_j(d) \) for all \( j \in T \), \( \bigcap_{i \in T} S_{(t)} = \{ d \} \). This means that there is at least one player \( k \) for which \( \text{payoff}_j(d_i) > \text{payoff}_j(d) \) for every \( (d \neq j) \cap \{ S \} \). Therefore, for every \( (d \neq j) \cap \{ S \} \), there is a player \( k \) such that \( \text{payoff}_j(d_i) > \text{payoff}_j(d) \), which implies that \( \text{DIS}_k(d) \leq \text{max} \{ \text{DIS}_k(d) \} \) \( \leq \text{DIS}_k(d) \) \( \leq \text{DIS}_k(d) \). Hence, \( \text{DIS}_k(d) < \text{DIS}_k(d) \) for all \( k \) and thus \( \text{max} \{ \text{DIS}_k(d) \} < \text{DIS}_k(d) \). In conclusion, \( d \) minimizes the maximum dissatisfaction of all players. \( \blacksquare \)

Appendix C: Discussion on NE provided by Saeed and Larsen (2010a, 2013)

Saeed and Larsen (2010a, 2013) assumed that \( \theta \left( \frac{x}{\text{CAP}} \right) = \epsilon \) which is a constant. Suppose that profit function (7) is used. The NE is characterized by the first-order conditions of Eq. (C1).

\[
\delta \frac{\prod_x}{\partial p_i} = x_i + (p_i - w_i - c_i) \frac{\partial x_i}{\partial p_i} \tag{C1}
\]

By taking the log of Eq. (6),

\[
\ln(x_i) = \ln(Ae^{eLS}Q_i) = \ln(A) + gLS + U_i - LS.
\]

(C2)

By taking the derivative of the above Eq. (C2) with respect to \( p_i \),

\[
\frac{\delta \ln(x_i)}{\partial p_i} = \frac{\delta(x_i)}{\partial p_i} \frac{1}{x_i} = \frac{\theta(Ae^{eLS}Q_i)}{\partial p_i} \frac{1}{\theta(Ae^{eLS}Q_i)} + \frac{\theta(U)\theta(LS)}{\partial p_i} - \frac{\theta(LS)}{\partial p_i}.
\]

(C3)

The derivative of \( x_i \) with respect to \( p_i \) can be expressed as

\[
\frac{\delta x_i}{\partial p_i} = Ae^{eLS}Q_i \left( \frac{\theta(LS)}{\partial p_i} + \frac{\theta(U)\theta(LS)}{\partial p_i} - \frac{\theta(LS)}{\partial p_i} \right)
\]

(C4)

Let us calculate the derivative of \( LS \) and \( U_i \) with respect to \( p_i \), respectively. Eqs. (C5) and (C6) can be derived.

\[
\frac{\delta(LS)}{\partial p_i} = \frac{\delta \left( \ln \left( \sum_j e^{j} \right) \right)}{\partial p_i} = \frac{1}{\sum_j e^{j}} \frac{\delta \left( \sum_j e^{j} \right)}{\partial p_i} = \frac{1}{\sum_j e^{j}} \frac{\delta(e^{j})}{\partial p_i} + \frac{1}{\sum_j e^{j}} \left( \sum_{j \neq i} \frac{\delta(e^{j})}{\partial p_i} \right)
\]

(C5)

\[
\frac{\delta(U)\theta(U)}{\partial p_i} = b + \frac{\delta(OUC)}{\partial p_i}
\]

(C6)

Saeed and Larsen (2010a, 2013) derived \( \frac{\delta(LS)}{\partial p_i} = b \frac{\delta(e^{i})}{\partial p_i} \) by approximating \( \frac{\delta(e^{j})}{\partial p_i} = 0 \) and \( \frac{\delta(U)}{\partial p_i} = b + \frac{\delta(OUC)}{\partial p_i} = b \), which are not true for the following reason. The volume handled by a container terminal depends on the charge per TEU in the container terminal. The relationship between \( x \),
and $p_i$ is shown in Eq. (6). Hence, $x_i$ and $p_i$ are dependent variables. Therefore, $\frac{\partial (Q(x_i))}{\partial x_i} \neq 0$ and for the same reason, $\frac{\partial (Q(x_i))}{\partial p_i} \neq 0$. By this simplification, they derived a simple equation system: $p_i = \tilde{A}_i(Q_i)$, for all terminal $i$, which can be solved by a numerical method. The size of the errors from this simplification by Saeed and Larsen (2010a, 2013) are evaluated by numerical experiments in Section 5.3.

References


