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PICKUP HUB LOCATION PROBLEM WITH PROSPECTIVE CUSTOMERS IN RURAL AREA: A CASE STUDY

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This paper addresses the pickup hub location problem in rural areas, considering dynamic changes in prospective customers and optimizing equity. The problem involves selecting optimal pickup hub locations from candidate hubs while minimizing transportation costs, measured as the distance between a set of determined customers and pickup hubs. A nonlinear fractional integer programming model is developed to formulate the problem. A scenario-based Dinkelbach's algorithm combined with a mathematical reformulation approach is proposed to solve the problem efficiently. The effectiveness of the proposed method is demonstrated through a case study on the location selection of smart cigarette delivery lockers. The results highlight the method's ability to balance equity, offering a practical solution for logistics planning in rural areas. The key contributions of this study are: (1) a novel pickup hub location model that accounts for dynamic customer changes and (2) validation of the approach through a real-world case study, showcasing its applicability and effectiveness.

Keywords: Mathematical modeling, Fractional Programming in Logistics, Pickup Hub Optimization, equity, Dinkelbach's Algorithm for Facility Location

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1. BACKGROUND

Doorstep delivery service, which is widespread in cities, involves delivering products to customers (Shi *et al*., 2023; Brunetti *et al*., 2024; Zhou *et al*., 2025). A pickup hub location, also known as a pickup point, is a facility that distributes products ordered online by customers. In cities, pickup stations are typically located very close to customers. These locations are often conveniently located in retail stores, shopping centers, or designated pickup points. A study by Lee *et al*. (2023) found that doorstep delivery services could help communities improve health conditions, reducing hospital visits and lowering medical expenses compared to communities without doorstep public service. Accessing the pickup hub location is increasingly important for governments and companies providing equity services.

According to the news reported by Luo (2023), approximately 95 % of Chinese villages now have access to pickup stations. However, in rural areas, the distance between the pickup station and the customer's home is much farther than that of metropolitan pickup stations. To promise more doorstep delivery services in rural areas, the national and local governments plan to invest vast capital to expand the rural delivery network. As highlighted in previous studies, cost is a critical factor when selecting pickup hubs. The local government requires logistics companies to provide last-mile delivery services in rural areas to ensure equitable last-mile delivery services. Three or four villages share at least one pickup hub. Figure 1 shows an example of a pickup hub servicing three villages. In this example, we can find that the pickup hub is located in village 1.

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Customers located in villages 2 and 3 should go to village 1 to get their orders. Due to the customers being randomly distributed and simply for the calculation, we could use the candidate pickup hubs among different villages to represent the distance between customers and the pickup hubs. Equity has been widely studied in relief routing problems by many previous studies (Huang *et al*., 2012b; Gu *et al*., 2018; Zhou and Lee, 2020). In rural China, customers get their orders by themselves. Hence, equality in accessing their orders needs to be considered when determining the pickup hubs.

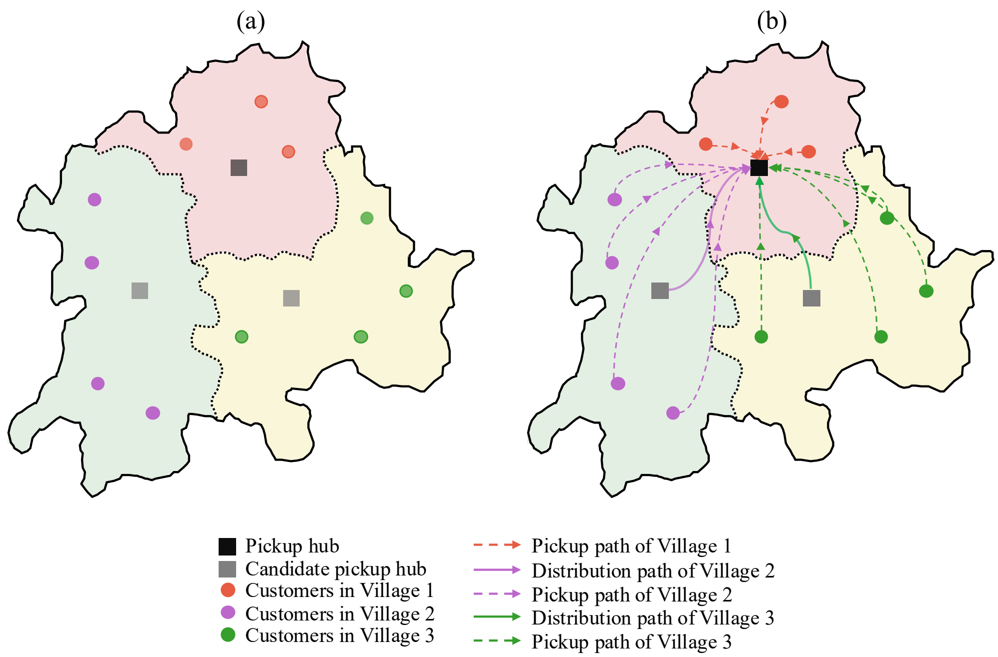


Figure 1. An example of a pickup hub serving three villages: illustration of location and customer access.

In previous studies of pickup hub location problems, there was a set of candidate pickup hub locations and a fixed number of customers. Many logistics companies provide the service of joining a courier franchise to open a pickup hub. Usually, convenience stores are selected as candidate pickup hubs. The total cost of opening a pickup hub is the same in such a case. The customers in rural areas may change over a specific period. Despite the growing importance of pickup hubs in rural areas, existing models for facility location problems often fail to account for prospective customers—those whose demand may emerge or change over time. Most studies assume a fixed set of customer locations and demands, which does not reflect the dynamic nature of rural communities. For example, customer distribution and demand patterns in rural areas can change due to population migration, economic development, or seasonal variations. Ignoring these dynamics can lead to suboptimal hub placement, inefficient resource allocation, and reduced service quality. Considering the above limitation in the current pickup hub location problem, the contributions of this paper are summarized as follows: In this paper, we study the pickup hub location problem considering the prospective customers by minimizing the maximum average distances from the pickup hub to customers among all the setup pickup hubs. A discrete scenario-based nonlinear fractional integer model is developed to formulate the studied problem. To solve the discrete scenario-based nonlinear fractional integer model, Dinkelbach's algorithm and reformulation method are proposed to solve the studied problem. A case study is conducted to illustrate the studied problem.

The positioning of this paper within the existing research framework is as follows：

Hub Location Models: Our study departs from traditional hub location models. While past models often focus on static customer bases and cost-only optimization, we account for the dynamic nature of rural customer populations. For example, existing models like the p-median model (Hakimi, 1964) optimize location based on a fixed set of demand points, which is ill-suited for rural areas with changing customer demands. Our model incorporates prospective customers, thus adapting to the evolving rural environment.

Equity-Based Optimization: In the context of equity-based optimization, prior research has mainly focused on urban-centric applications or simple resource-sharing scenarios. Our study extends this to rural last-mile delivery, where equity in access to pickup hubs is crucial due to the self-collection nature of rural customers. By minimizing the maximum average distance from pickup hubs to customers, we ensure fair access across different villages in rural delivery equity optimization.

Scenario-Based Decision-Making: Existing scenario-based decision-making in facility locations often uses static scenarios. In contrast, our discrete scenario-based model accounts for various dynamic factors in rural areas, such as population migration and economic development. These scenarios are integrated into our model to make more robust decisions regarding pickup hub locations.

The remainder of this paper is organized as follows. Section 2 summarizes the previous studies related to this paper. Section 3 presents the pickup hub location problem model considering equity efficiency. Section 4 gives methods for solving the studied problem. A case study is provided in section 5. Finally, the conclusions are presented in Section 6.

2. LITERATURE REVIEW

The literature has extensively studied facility location problems, with various models proposed to address different objectives and constraints. These problems are widely applied in determining the optimal locations of facilities such as hospitals, warehouses, emergency response facilities, and fire stations. Existing models can be broadly categorized into classical models and extended models, each focusing on different optimization objectives and constraints. Classical facility location models primarily focus on optimizing single objectives, such as cost minimization or coverage maximization. Some of the most widely studied classical models include the p-median problem, p-center problem, maximum covering location problem, etc.

The min-max facility location problem is a classical p-center model that has been studied for a long time. The location problem is usually used to determine the location of facilities, such as hospitals, warehouses, emergency response facilities, and fire stations. Alumur and Kara (2008) conducted a systematic review of network hub location research, focusing on optimizing traffic transmission between source and destination by selecting hub locations and allocating demand nodes. They highlighted the significant growth of this field in recent years. They pointed out that although the research on the P-hub median problem is mature, other problems, such as the hub center and coverage problem, still need more theoretical and algorithmic development. Farahani *et al.* (2013) comprehensively reviewed hub location problems, discussing models, classifications, solutions, and applications. This paper examines the role of hubs as transfer stations for people, goods, or information from origin to destination and analyzes the latest progress in this field since 2007. In addition, the research trends in this field are pointed out, such as multi-objective optimization, reliability modeling, and design of global logistics networks.

One of the popular objective functions is to minimize the maximum Euclidean distance between given facilities and customers. Such as $\text{Min}\left(\max_{1\le i\le n}\{d(v_i,p_j)\}\right)$ where v_i represents the demand points and p_j represents the given facilities. $d(v_i,p_j)$ is the Euclidean distance function between v_i and p_j. This model is used for a facility location problem where a single facility serves multiple customers. The problem is locating a single facility or multiple facilities so that the maximum distance between the given facility and the demand customer points is minimized. Zhang (2021) proposed two minimax models to minimize the maximum distances from the facility location to customers. These models are linear integer programming models (Dolu, 2020; Elloumi, 2004).

Another objective function is to minimize the maximum weighted distance of the one-center location problem or one facility min-max location problem. This model is as follows: $\text{Min}\left(\max_{i=1,\dots,m}\{\omega_i||p_i-X||\}\right)$. ω_i is the weight. In the model, X is a new point chosen from a set of given different points. p_i is the existing facility location, and $||p_i-X||$ is the Euclidean distance between p_i and X. For example, P.M.Dearing (1974) used this model to minimize the maximum of linear increasing functions of distances between the new facility and the existing facilities. Many studies adopted the maximum weighted distance as an objective function (Lin, 2010; Chandrasekaran, 1980).

There is another model that minimizes the maximal total length of a vehicle's tour in a traditional vehicle routing problem with a single depot (Albareda-Sarnbola, 2019; Jiang, 2012; Du, 2020), such as $\text{Min}\left(\max_{\forall i}\{\sum_{j\in J} d_{ij} * x_{ij}\}\right)$. d_{ij} is the length of the depot j serviced by vehicle i and x_{ij} is binary variables, which $x_{ij}=1$ means depot j serviced by vehicle; otherwise, $x_{ij}=0$. Narasimha *et al*. (2013) applied this model to the multi-depot vehicle routing problem, aiming to minimize the maximum distance traveled by any vehicle rather than the total distance.

In the facility location problem, uncertainty arises in the demand and service time. To address these problems with uncertain parameters, the standard method is to optimize the worst-case performance. Wu (2020) optimized the worst observation of all scenarios based on the worst-case criterion. Baldomero-Naranjo (2021) proposed a min-max regret model for uncertainty with interval estimation. Ataei (2023) considered scenarios where demand point locations may change within a region. In this paper, uncertainty arises in the number and location of the prospective customer centers. To solve this problem, we consider all the scenarios based on the uncertainty and minimize the worst-case performance of the scenarios.

Based on previous research results, this paper introduces distance fairness between pickup hubs and their serviced customers and further enhances the model's practical applicability. In addition, the paper subdivides the customers, including the determined customer centers and the prospective customer centers. This paper's model more accurately reflects the actual situation and provides more targeted strategies for optimizing the program. These improvements make the model more flexible and efficient in solving practical problems and help to improve the fairness and overall performance of logistics network design. This paper minimizes the maximum average distance between pickup hubs and their serviced customers. The min-max model primarily addresses geographical equity issues, such as fire, police, and medical ambulance services. $\mathrm{Min}\left(\max_{i\in P, x_i=1}\left\{\frac{(\sum_{i\in P} c_{ij}*y_{ij})}{\alpha+\sum_{i\in P} x_{ij}}\right\}\right)$ is the objective function, which is a nonlinear fractional function. The average distance between pickup hubs and their serviced customers is mainly applied in cases where the distance to each point is critical. 0 summarizes the related studies.

Table 1. Summary of the related studies.

Studies	Goal	Model type	Equity	Location Uncertainty	Solution method
Zhang (2021)	Min-max distance	UP	✓		AHA
Dolu (2020)	Min-max distance	SOCP	✓		MFMM
Elloumi (2004)	Min-max distance	IP	✓		PA
P.M.DEARING (1974)	Min-max weighted distance	LP			NSA
Lin (2010)	Min-max weighted distance	GP			GA
Chandrasekaran (1980)	Min-max weighted distance	AP			PB
Narasimha *et al*., (2013)	Min max total distance	DP			ACO
Jiang (2012)	Min-max is the sum of weighted distances	IP			PCM
Albareda-Sarnbola (2019)	Min the weighted average of the largest distances	IP			SAA
Du (2020)	Min-max the sum of weighted distances	IP			LIR, BDPM, CCGA
Wu (2020)	Min-max total time	MILP			HA
Baldomero-Naranjo (2021)	Min-max regret	NLP			PTA
Ataei (2023)	Min-max regret	NLP		✓	AA
This paper	Min-max average distance	FFP	✓	✓	DA

[1]UP: Uncertain programming; SOCP: Second order cone programming; IP: Integer programming; LP: Linear programming; GP: Geometric Programming; QP: Quadratic programming; DP: Dynamic programming; MILP: Mixed-integer linear programming; NLP: Non-Linear programming FP: Fractional programming

[2]AHA: A hybrid algorithm MFMM: MISOCP formulation, Minkowski method, PA: Polynomial Algorithm, NSA: new, simple algorithm; GA: Genetic algorithm; PBA: Polynomially bounded algorithm, ACO: Ant Colony Optimization, PCM: Projection contraction method, SAA: Sample Average Approximation, LIR: Linear integer reformulation, BDCPM: Bender's dual cutting plane method, CCGA: Column-and-constraint generation algorithm, PTA: Polynomial-time algorithm, AA: Approximation algorithms, DA: Dinkelbach's algorithm

Gao (2015) conducted experimental comparisons on several methods for solving fractional programming problems, including the proposed Branch-and-Bound algorithm, the parametric algorithm (Dinkelbach's algorithm), the reformulation-linearization method, as well as general MINLP solution methods such as DICOPT (outer approximation), SBB (simple Branch-and-Bound algorithm), and global optimizers BARON 12 and SCIP 3. The results indicate that the parametric algorithm (Dinkelbach's algorithm) and the reformulation-linearization method outperform other solution methods. Therefore, this paper selected Dinkelbach's algorithm and the reformulation-linearization method for model solving. Compared to the reformulation-linearization method, Dinkelbach's algorithm does not introduce new auxiliary variables, and its mathematical structure is straightforward, demonstrating efficiency and convergence in handling fractional optimization problems.

Previous research has primarily focused on efficiency optimization, often neglecting equity issues. In contrast, this paper introduces fairness as a core objective based on the traditional min-max model. It proposes a nonlinear fractional integer programming model to minimize the maximum average distance between pickup hubs and their serviced customers. Additionally, this paper incorporates dynamic customer demands and uncertainties, optimizing worst-case performance through scenario analysis, further enhancing the model's practicality and robustness.

3. A DISCRETE SCENARIO-BASED MODEL CONSIDERING PROSPECTIVE CUSTOMERS

3.1 Problem Statement

Before introducing details of the studied problem, the notations, parameters, and decision variables are shown in Table 2.

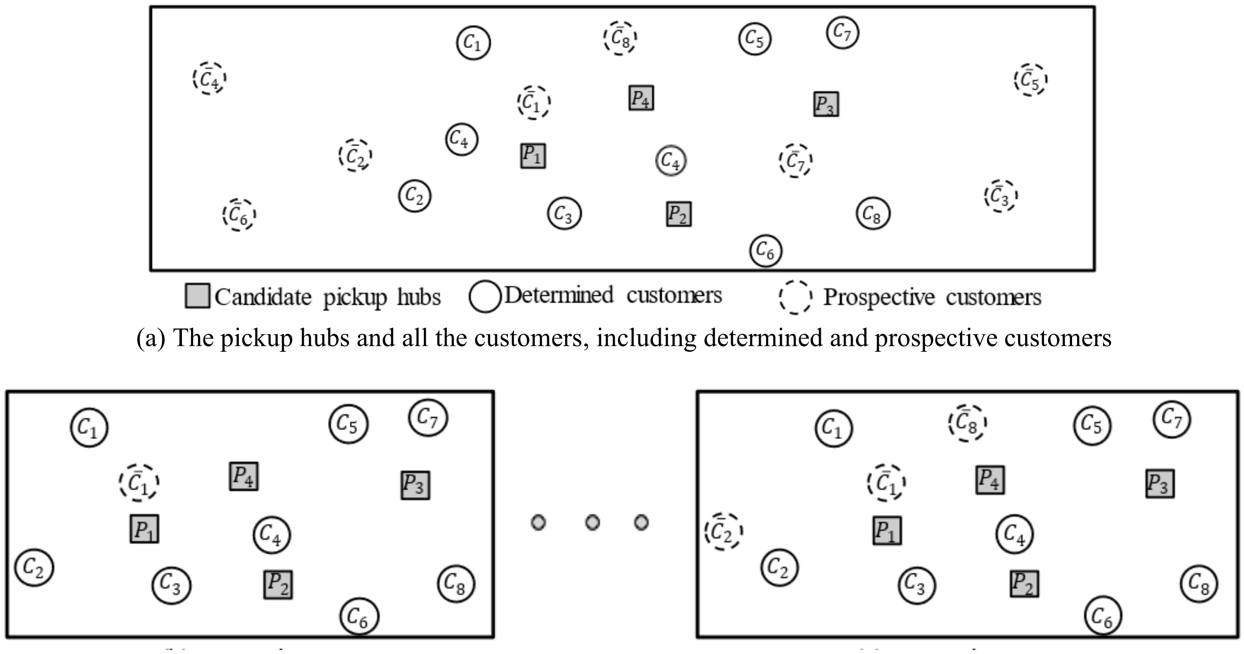
Table 2. Notations, parameters, and decision variables.

Set	Meaning
S	Set of scenarios
$\tilde{C}$	Set of determined customer centers
$\bar{C}_s$	Set of prospective customer centers in scenarios
C_s	Set of customer centers in scenario s, $C_s = \tilde{C} \cup \bar{C}_s$
C	Set of customer centers, $C = \tilde{C} \cup \bar{C}_s \cup \ldots \bar{C}_{\lvert S \rvert}$
P	Set of pickup hubs
Index	**Meaning**
i	Index of pickup hub
j	Index of customer center
s	Index of scenario
Parameters	**Meaning**
p	Maximum number of customer centers serviced by a pickup hub
α	A very small real number
γ	Total number of pickup hubs to be constructed
M	A very large value
c_{ij}	The distance between pickup hub i and customer j
Decision variable	**Meaning**
x_i	If the pickup hub is set up, $x_i = 1$. Otherwise, $x_i = 0$.
y_{ij}	If customer center j is serviced by pickup hub i, $y_{ij} = 1$. Otherwise, $y_{ij} = 0$

A local government wants to set up pickup hubs (P) to provide public delivery services in rural areas. In these regions, customers are widely distributed, and their loyalty may fluctuate over time. Let C denote the set of determined customer centers, which means that these customer centers will always use public delivery services for a very long period. Additionally, there is a set of prospective customer centers that may adopt public delivery services in the near future. In rural areas, customers typically travel to pick-hubs to get their orders. Many companies in rural China do not provide last-mile door-to-door service. Hence, each pickup hub could only service a certain number of customer centers, and p is the maximum number of customer centers serviced by a pickup hub. To ensure fairness, the average service distance from each pickup hub to its customer centers is minimized as much as possible. Based on the principle of fairness, the objective is to minimize the maximum average delivery distance from the pickup hub to the customer centers, considering the prospective customer centers. Let S denote the set of scenarios and $\bar{C}_s$ is the set of prospective customer centers in scenario s. $\tilde{C}$ is the set of determined customer centers in scenario s and $C_s = \tilde{C} \cup \bar{C}_s$.

Figure 2 shows an example of customer and pickup hub distributions across different scenarios. Figure 2a shows the pickup hubs and all the customers, including determined and prospective customers. Rectangles represent pickup hubs, circles represent the determined customer centers, and dotted circles represent the prospective customer centers. Figure 2b and Figure 2c show two distinct scenarios. Meanwhile, Figure 2b illustrates a scenario where a potential customer may appear at the location $\bar{C}_1$, representing a specific hypothetical situation. Figure 2c depicts a scenario where multiple potential customers appear at location $\bar{C}_1$, $\bar{C}_2$, and $\bar{C}_8$, further revealing the possible concentration of customers at these sites. These figures help visualize the complexity of potential customer distribution and its implications for location decision-making.

The assumptions used in this paper are summarized as follows: (1) Capacity constraints of pickup hubs are not considered. (2) The number of customer centers served by each pickup hub is limited. (3) Customers need to pick up their orders from their homes and take them to the pickup hubs by themselves. (4) All pickup hubs have identical construction costs. The case study considers smart cigarette delivery lockers as pickup hubs. The smart cigarette delivery lockers are homogeneous, and their costs are the same. (5) The model uses spherical distance for calculations.



(a) The pickup hubs and all the customers, including determined and prospective customers

(b) Scenario 1

(c) Scenario s

Figure 2. Distribution of customers and pickup hubs in different scenarios.

3.2 A Nonlinear Fractional Model

This paper adopts strict robustness to identify a robust solution that applies to all scenarios. The objective function (1) aims to minimize the maximum average distance cost for customers serviced by each pickup station, selecting the maximum value among the optimal solutions of all scenarios. The objective function (1) is a nonlinear fractional function.

$$P \qquad \mathrm{Max}_{s\in S} \quad \underbrace{\mathrm{Min} \quad \underbrace{\max_{i\in P}\left\{\underbrace{\frac{\sum_{j\in C}(c_{ij}*y_{ij})}{\sum_{j\in C} y_{ij}}}_{\text{Average distance of pickup hub j}}\right\}}_{\text{The maximum average distance between customers and pickup hub}}}_{\text{Minimize the maximum average distance between customers and pickup hub}} \tag{1}$$

subject to

$$\sum_{i\in P} y_{ij} = 1 \quad \forall \quad j \in C \tag{2}$$

$$\sum_{i\in P}^{n} x_i = \gamma \quad \forall \quad i \in P \tag{3}$$

$$\sum_{j\in C} y_{ij} \le p \quad \forall \quad i \in P \tag{4}$$

$$y_{ij} \le x_i \qquad \forall \quad i \in P, j \in C \tag{5}$$

$$x_i \in \{0,1\} \quad \forall \quad i \in P, j \in C \tag{6}$$

$$y_i \in \{0,1\} \quad \forall \quad i \in P, j \in C \tag{7}$$

Constraint (2) ensures that each customer center is serviced by only one pickup station. Constraint (3) denotes that the total number of opened pickup stations is γ. Constraint (4) represents the upper limit on the number of customer centers that a pickup station can serve. Constraint (5) shows that a customer center can only be assigned to an open pickup station. Constraints (6) and Constraint (7) represent that x_i and y_{ij} are binary decision variables.

3.3 Sub-Problem

In this subsection, we analyze the mathematical model of each scenario, which is a sub-problem for the studied problem.

3.3.1 Mathematical Model of Each Scenario

$$P_s^0 \qquad \text{Min} \quad \max_{i \in P} \left\{ \frac{\sum_{j \in C} (c_{ij} * y_{ij})}{\sum_{j \in C} y_{ij}} \right\} \tag{8}$$

subject to

$$\sum_{i \in P} y_{ij} = 1 \qquad \forall \quad j \in C_s \tag{9}$$

$$\sum_{i \in P}^{n} x_i = \gamma \qquad \forall \quad i \in P \tag{10}$$

$$\sum_{j \in C_s} y_{ij} \leq p \qquad \forall \quad i \in P \tag{11}$$

$$y_{ij} \leq x_i \qquad \forall \quad i \in P, j \in C_s \tag{12}$$

$$x_i \in \{0,1\} \qquad \forall \quad i \in P, j \in C_s \tag{13}$$

$$y_{ij} \in \{0,1\} \qquad \forall \quad i \in P, j \in C_s \tag{14}$$

The constraints of this model are structurally consistent with those of the model P, with the primary difference being in the definition of sets.

Theorem 1. The objective function (8) of the relaxed problem of the problem P^0 *is both pseudoconvex and pseudoconcave.*

Proof. See chapter 11.4 of Bazaraa *et al*. (2013)

To prevent the denominator of the objective function (8) from being zero, α is added to the denominator, where α is a very small number. The objective function (8) is expressed as the objective function (15).

$$P_s^1 \qquad \text{Min} \quad \max_{i \in P} \left\{ \frac{\sum_{j \in C} (c_{ij} * y_{ij})}{\alpha + \sum_{j \in C} y_{ij}} \right\} \tag{15}$$

Subject to (9)-(13).

3.3.2 Linearization Of The Sub-Problem

Model P_s^1 could not be solved directly using integer programming solvers, such as IBM Cplex, Gurobi, Mosek, and Lingo. Inspired by the study of Yue *et al*. (2013), we convert the nonlinear fractional model P_s^1 to a nonlinear integer programming model by introducing two auxiliary variables μ_j, h_{jk}, defined as follows.

$$\mu_i = \frac{1}{\alpha + \sum_{j' \in C_s} y_{ij'}} \qquad \forall \quad i \in P \tag{16}$$

$$h_{ij} = \frac{y_{ij}}{\alpha + \sum_{j' \in C_s} y_{ij'}} \qquad \forall \quad i \in P, j \in C_s \tag{17}$$

Therefore, Min $\max_{i \in P} \left\{ \frac{\sum_{j \in C} (c_{ij} * y_{ij})}{\alpha + \sum_{j \in C} y_{ij}} \right\}$ = Min $\max_{i \in P} \left\{ \sum_{j \in C} c_{ij} * \frac{y_{ij}}{\alpha + \sum_{j \in C} y_{ij}} \right\}$ = Min $\max_{i \in P} \{ \sum_{j \in C} c_{ij} * h_{ij} \}$, the proposed nonlinear integer programming model is presented in P_s^2.

$$P_s^2 \qquad \text{Min} \quad \left(\max_{i \in P} \left\{ \sum_{j \in C_s} c_{ij} * h_{ij} \right\} \right) \tag{18}$$

subject to (9)-(13) and

$$h_{ij} = y_{ij} * \mu_i \qquad \forall \quad i \in P, j \in C_s \tag{19}$$

$$\sum_{j \in C_s} h_{ij} + \alpha * \mu_i = 1 \qquad \forall \quad i \in P, j \in C_s \tag{20}$$

$$\mu_i \geq 0 \qquad \forall \quad i \in P, j \in C_s \tag{21}$$

$$h_{ij} \geq 0 \qquad \forall \quad i \in P, j \in C_s \tag{22}$$

For Constraint (19), $h_{ij} = \frac{y_{ij}}{\alpha + \sum_{j \in C_s} y_{ij}} = y_{ij} * \frac{1}{\alpha + \sum_{j \in C_s} y_{ij}} = y_{ij} * \mu_i$. We include the Constraint (20) in the reformulation to define the variable, $\frac{1}{\alpha + \sum_{j \in C_s} y_{ij}} * \left(\alpha + \sum_{j \in C_s} y_{ij} \right) = \frac{\sum_{j \in C_s} y_{ij}}{\alpha + \sum_{j \in C_s} y_{ij}} + \frac{\alpha}{\alpha + \sum_{j \in C_s} y_{ij}} = \sum_{j \in C_s} h_{ij} + \alpha \mu_i = 1$. For constraints (21) and (22), y_{ij} is binary variables, therefore, $\mu_i \geq 0$, and $h_{ij} \geq 0, \forall \quad i \in P, j \in C_s$.

The model P_s^2 is a nonlinear integer programming model with a nonlinear $h_{ij} = y_{ij} * \mu_i$. To linearize the nonlinear Constraint $h_{ij} = y_{ij} * \mu_i$, we introduce the following constraints.

$$h_{ij} = \begin{cases} 0, & \text{if } y_{ij} = 0 \\ \mu_i, & \text{otherwise} \end{cases} \tag{23}$$

$$h_{ij} \leq \mu_i \qquad \forall \quad i \in P, j \in C_s \tag{24}$$

$$h_{ij} \leq M y_{ij} \qquad \forall \quad i \in P, j \in C_s \tag{25}$$

$$h_{ij} \geq \mu_i - M(1 - y_{ij}) \qquad \forall \quad i \in P, j \in C_s \tag{26}$$

Formulation (23) implies that $h_{ij} = \mu_i$ or 0, in Constraint (25), M is a sufficiently large number, and Constraint (25) implies that if y_{ij} is zero, then h_{ij} should be zero; constraints (24) and (26) indicate that if y_{ij} is one, then h_{ij} should be equal to μ_i. Thus, constraints (24) - (25) are linearization constraints for $h_{ij} = y_{ij} * \mu_i$.

Theorem 2. The optimal solution (X^*, Y^*) of the problem P_s^1 is an optimal solution of the P_s^0.
Proof. (A) (X^*, Y^*) is a feasible of the problem P_s^0. (B) (X^*, Y^*) is the optimal of the problem P_s^0,. See Li (1994)

3.3.3 An Example

This section is a computational study of a small example. The test case is that a tobacco company has an order for 8 cigarette demand points in a certain area. To fulfill this order, the company plans to build 2 smart cigarette delivery lockers in the area.

The objective is to minimize the maximum average distance from the smart cigarette delivery locker to customers, ensuring equitable access to the delivery services. This example assumes 4 candidate pickup hub locations, as shown in Figure 3.

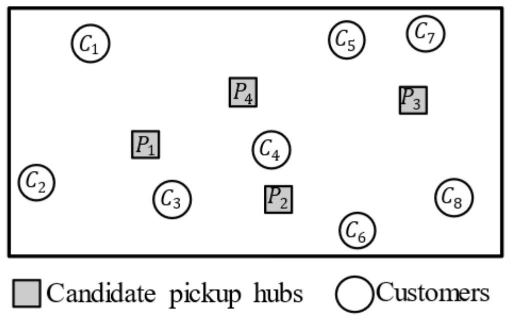


Figure 3. Distribution of customers and candidate pickup hubs in a Tobacco company's order.

Figure 3 shows the distribution of customers and smart cigarette delivery lockers. The solid squares represent the candidate smart cigarette delivery locker locations, and the circles represent customers.

Table 3. The distance between customers and smart cigarette delivery lockers.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
P_1	4	2	3	6	18	14	20	24
P_2	12	10	4	5	12	2	30	12
P_3	20	25	16	9	7	4	2	6
P_4	6	10	14	2	3	9	11	22

Table 3 shows the distance between customers and the candidate smart cigarette delivery lockers; P represents candidate smart cigarette delivery lockers, and C represents customers.

Table 4. Optimal solutions for the example.

Smart Warehouse	Demand Point
P_1	C_1, C_2, C_3, C_4
P_3	C_5, C_6, C_7, C_8
optimal value	4.749

We used the model P_S^2 to formulate the studied problem, and it is solved by the IBM CPLEX solver. The obtained optimal solution is shown in the following table.

From Table 3 and Figure 4, the optimal smart cigarette delivery lockers are determined to be P_1 and P_3. The smart cigarette delivery locker P_1 services customers C_1, C_2, C_3 and C_4, while the smart cigarette delivery locker P_3 services C_5, C_6, C_7 and C_8. The optimal solution achieves a value of 4.749.

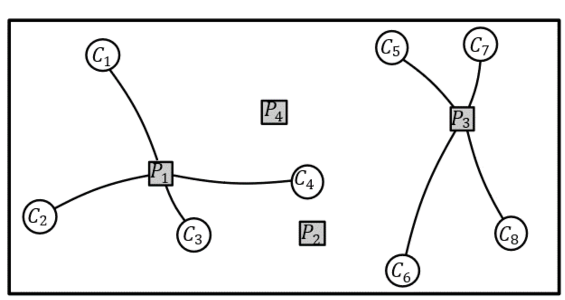


Figure 4. Visualization of the smart cigarette delivery lockers' location and allocation of the example.

3.4 Reformulation Of The Studied Problem

By using the linearized model of P_s^2 presented in section 3.3.2, the nonlinear fractional model P can be reformulated as P^R, which is a linear integer programming model that can be solved by solvers, such as IBM CPLEX.

$$P^R \qquad \text{Max}_{s\in S} \left(\text{Min} \left(\max_{i\in P} \left(\sum_{j\in C_s} c_{ij} * h_{ij} \right) \right) \right) \tag{27}$$

subject to

$$h_{ij} \le \mu_i \qquad \forall \quad i \in P, j \in C_s \tag{28}$$

$$h_{ij} \le M y_{ij} \qquad \forall \quad i \in P, j \in C_s \tag{29}$$

$$h_{ij} \ge \mu_i - M(1 - y_{ij}) \qquad \forall \quad i \in P, j \in C_s \tag{30}$$

$$\textstyle\sum_{j\in C_s} h_{ij} + \alpha\mu_i = 1 \qquad \forall \quad i \in P, j \in C_s \tag{31}$$

$$\textstyle\sum_{i\in P} y_{ij} = 1 \qquad \forall \quad j \in C_s \tag{32}$$

$$\textstyle\sum_{i\in P}^{n} x_i = \gamma \qquad \forall \quad i \in P \tag{33}$$

$$y_{ij} \le x_i \qquad \forall \quad i \in P, j \in C_s \tag{34}$$

$$\textstyle\sum_{j\in C_s} h_{ij} \le p * \mu_i \qquad \forall \quad i \in P \tag{35}$$

$$\mu_i \ge 0 \qquad \forall \quad i \in P, j \in C_s \tag{36}$$

$$h_{ij} \ge 0 \qquad \forall \quad i \in P, j \in C_s \tag{37}$$

$$x_i \in \{0,1\} \qquad \forall \quad i \in P, j \in C_s \tag{38}$$

$$y_{ij} \in \{0,1\} \qquad \forall \quad i \in P, j \in C_s \tag{39}$$

4. SOLUTION METHODS

Robust optimization is very important in facility location and routing problems (Shi *et al*., 2020). Subsection 4.1 presents Dinkelbach's algorithm for finding the optimal solution for each scenario. This paper introduces two methods for obtaining the strict robustness solution. Subsection 4.2 introduces the scenario based on Dinkelbach's algorithm, and Subsection 4.3 presents the scenario based on the mathematical formulation algorithm, respectively.

4.1 Dinkelbach's Algorithm For Solving Model P_s^0

P_s^0 is a multiple ratio nonlinear integer model, which can not be solved by solvers such as Cplex, CBC, Gurobi, LINGO, or Mosek. After linearization P_s^0, the resulting model P_s^2 becomes a linear mixed integer programming model, which can be solved by these solvers. However, the linearization process requires additional constraints, which makes the model P_s^2 time-consuming. In this section, we try to directly solve P_s^0 using Dinkelbach's algorithm, originally proposed by Dinkelbach (1967), for solving single ratio nonlinear fractional models. Ferland and Potvin (1985) extended Dinkelbach's algorithm for solving multiple ratio nonlinear fractional models. R´odenas *et al*. (1999) extended Dinkelbach's algorithm for solving integer fractional models.

$N_i(y_i) = \sum_{j \in C_s} c_{ij} * y_{ij}, D_i(y_i) = \alpha + \sum_{j \in C_s} y_{ij}$, where $y_i = (y_{i1}, \cdots, y_{ij}, \cdots, y_{i|C_s|})$.

The model P^1 is a nonlinear fractional programming model. We define $N_i(y_i) = \sum_{j \in C_s} c_{ij} * y_{ij}$ and $D_i(y_i) = \alpha + \sum_{j \in C_s} y_{ij}$, where $y_i = (y_{i1}, \cdots, y_{ij}, \cdots, y_{i|C_s|})$. By using $N_i(y_i)$ and $D_i(y_i)$, we convert model P_s^1 to an integer programming model P_s^2. Note that model P_s^1 and model P_s^2 are not equivalent. By using Dinkelbach's Algorithm, the original problem can be transformed into the following problem. Model P_s^{DA} is defined as follows.

$$P_s^{DA} \quad \text{Min} \quad \left(\max_{i \in P}\{N_i(y_i) - \lambda * D_i(y_i)\}\right) \tag{40}$$

subject to

$$N_i(y_i) = \sum_{j \in C} c_{ij} * y_{ij} \quad \forall \quad i \in P \tag{41}$$

$$D_i(y_i) = \alpha + \sum_{j \in C_s} y_{ij} \quad \forall \quad i \in P \tag{42}$$

$$\sum_{i \in P} y_{ij} = 1 \quad \forall \quad j \in C_s \tag{43}$$

$$\sum_{i \in P}^{n} x_i = \gamma \quad \forall \quad i \in P \tag{44}$$

$$\sum_{j \in C_s} y_{ij} \le p \quad \forall \quad i \in P \tag{45}$$

$$y_{ij} \le x_i \quad \forall \quad i \in P, j \in C_s \tag{46}$$

$$x_i \in \{0,1\} \quad \forall \quad i \in P, j \in C_s \tag{47}$$

$$y_{ij} \in \{0,1\} \quad \forall \quad i \in P, j \in C_s \tag{48}$$

The model P_s^{DA} has the same constraints as the original model but a different objective function, which is formulation (49). When $F(\lambda) = 0$, this problem has a unique optimal solution, which is exactly consistent with the global optimal solution of the original model P_s^2 (You *et al.*, (2009)). Therefore, solving the model P_s^2 is equivalent to finding the root of the equation $F(\lambda) = 0$. The Dinkelbach algorithm's steps are as follows. Firstly, we set $\delta = 10^{-6}$.

Step 1: Choose arbitrary $(\bar{X}_k, \bar{Y}_k)$ and set $\lambda_k = \max_{1 \le i \in |P|} \left\{\frac{N_i(\bar{y}_i^k)}{D_i(\bar{y}_i^k)}\right\}$ (or set $\lambda_k = 0$), initialize k by setting $k = 0$.

Step 2: Solve the problem $F(\lambda_k) = \text{Min} \quad \left(\max_{i \in P}\{N_i(y_i^{k*}) - \lambda_k * (D_i(y_i^{k*})\}\right)$, and denote the optimal solution as y_i^{k*}

Step 3: If $|F(\lambda_k)| < \delta$(optimality tolerance), stop and output y_i^{k*} as the optimal solution; If $|F(\lambda_n)| < \delta$, let $\lambda_{k+1} = \max_{1 \le i \in |P|} \left\{\frac{N_i(y_i^{k*})}{D_i(y_i^{k*})}\right\}$ and go to Step2 to replace k with k+1 and λ_k with λ_{k+1}.

Through the above major steps of Dinkelbach's algorithm, this problem can be solved by finding the original problem's optimal solution by iteration and an updating procedure. This paper adopted general Dinkelbach's algorithm for solving the model P_s^2, which is summarized in Algorithm 1.

Algorithm 1: General Dinkelbach's algorithm for solving model P_s^0

```
Function GDA（C_s）
  k = 1;
  Initialize λ_n by using (a) or (b);
  (a)λ_k = 0;
  (b) Choose a feasible solution(X̄_k, Ȳ_k),  and λ_k = max_{1 ≤ i ∈ |P|} { N_i(ȳ_i^k) / D_i(ȳ_i^k) };
  while F(λ_k) ≤ β do
    (X_k^*, Y_k^*) = IPSolver(N_i(y_i) − λ_k * D_i(y_i))  //solving model P_s^DA
```

Algorithm 1: General Dinkelbach's algorithm for solving model P_s^0

8 $\lambda_{k+1} = \max_{1 \leq i \in |P|} \left\{ \frac{N_i(y_i^{k*})}{D_i(y_i^{k*})} \right\}$;
9 $k = k + 1$
10 **end**
11 **return** *Optimum Solution*;

Line 3 of Algorithm 1 initializes the λ_k. There are two methods for initializing λ_k. Method (a) in line 4 set $\lambda_k = 0$. In method (B) of line 5, a feasible solution is given and λ_k is initialized by using the feasible solution. $F(\lambda_k)$ is defined in equation (49). When $F(\lambda_k) = 0$, the model is optimized.

$$F(\lambda_k) = \text{Min} \quad \left(\max_{i \in P} \{ N_i(y_i^{k*}) - \lambda * (D_i(y_i^{k*}) \} \right) \tag{49}$$

β is a very small number in line 6 of Algorithm 1. In this paper, we set $\beta = 10^{-6}$. IPSolver() is an arbitrary integer programming solver, such as Cplex, CBC, Gurobi, LINGO, or Mosek. We can obtain the optimal solution (X_k^*, Y_k^*), where $X_k^* = (x_1^{k*}, \cdots, x_{|p|}^{k*})$, $Y_k^* = (y_1^{k*}, \cdots, y_{|p|}^{k*})$, and $y_i^{k*} = (y_{i1}^{k*}, \cdots, y_{ij}^{k*}, \cdots, y_{i|C_s|}^{k*})$ by using an arbitrary integer programming solver to solving the model P_s^2. By using equation (50) we can update the λ_{k+1}. This updating procedure is the key of the Dinkelbach's algorithm for solving model P_s^{DA}.

$$\lambda_{k+1} = \max_{i \in P} \left\{ \frac{N_i(y_i^{k*})}{D_i(y_i^{k*})} \right\} \tag{50}$$

4.2 Scenario-based Dinkelbach's Algorithm

Strict robustness is a mathematical concept in robust optimization that aims to find the worst-case solution among all the possible scenarios. For each scenario, we can apply Dinkelbach's algorithm to obtain the optimal solution, as described in section 4.1. After obtaining the optimal solutions for all scenarios, we compare them to find the worst solution, which is the strict robustness of the studied problem. By using the above process, we introduce the scenario based on Dinkelbach's algorithm, which is shown in Algorithm 2.

In algorithm 2, s represents the scenario s. In line 3, $RobusSolution$ represents the current obtained strict robustness solution. The while loop in line 5 iterates through all scenarios. $Solution$ in line 6 is the optimal solution of the scenario s. In line 7, if the optimal solution is worse than the current obtained strict robustness solution, we update the $RobusSolution$.

Algorithm 2: Scenario-based Dinkelbach's algorithm

1 **Function SDA** ( )
2 $s = 1$;
3 $RobusSolution = \emptyset$; // The worst solution among all the Scenario
4 $RobusObj = -\infty$; // Init the robust solution
5 **while** $s <= |S|$ **do**
6 $Solution = GDA(Cs)$; // the optimal solution of scenario s
// If find a worse solution, set it as the current robust solution
7 **if** $(ObjFun(Solution, Cs) > RobusObj)$ **then**
8 $RobusSolution = Solution$;
9 $RobusObj = ObjFun(Solution, Cs)$;
10 **end**
11 $s = s + 1$; // visit next scenario $s + 1$
12 **End**
13 **return** *Optimum Solution*; // return the robust solution

4.3 Scenario-based Mathematical Formulation Algorithm

The basic idea of the scenario-based mathematical formulation algorithm is similar to scenario-based Dinkelbach's algorithm. The key difference is that the scenario-based mathematical formulation algorithm uses a mathematical formulation to obtain the optimal solution for each scenario s. In line 6, the $MipSlover()$ is a mixed integer solver, such as CPLEX. P_s^2 is the mathematical model of scenario s that could be directly solved by the $MipSlover()$. The remaining parts of Algorithm 3 are the same as those of Algorithm 2.

Algorithm 3: Scenario-based mathematical formulation algorithm.

1	**Function SMFA** ( )
2	$s = 1$;
3	$RobusSolution = \emptyset$;
4	$RobusObj = -\infty$;
5	**while** $s <= \lvert S \rvert$ **do**
6	$(Solution, Objs) = MipSlover(P_s^2)$; // *MipSlover* is a mixed integer solver / / if find a worse solution, set it as the current robust solution
7	**if** $(ObjFun(Solution, Cs) > RobusObj)$ **then**
8	$RobusSolution = Solution$;
9	$RobusObj = ObjFun(Solution, Cs)$;
10	**end**
11	$s = s + 1$;
12	**End**
13	**return** *Optimum Solution*;

4.4 Implementation Environment

All the mathematical models proposed in this paper are solved by using IBM ILOG CPLEX Optimization Studio (version 12.9) with a Python interface to the CPLEX callable library. The Dinkelbach algorithm is implemented in the Python language (version 3.8.8). Both the models and algorithms are tested on a Windows 11 operating system with AMD Ryzen 5 5500U with Radeon Graphics 2.10 GHz and 16 GB of RAM.

5. A CASE STUDY

This section introduces a case study and compares the two proposed methods. Sensitive analysis and management insights are also provided. The details are shown as follows.

5.1 Cigarette Delivery Locker

In this section, the cigarette delivery locker is considered the pickup hub. The cigarette is franchised in China. A Local Tobacco Monopoly Bureau will distribute cigarettes to retail customer stores in the last mile of the cigarette distribution service. The quality of the 'last mile' delivery service significantly impacts customer satisfaction. If customers are dissatisfied with the service, their cigarette ordering frequency may decrease. Cigarette sales endpoint has dispersion characteristics, a large number, a wide distribution, and a large demand. To enhance distribution efficiency and meet the timeliness of cigarette products, tobacco companies further integrate logistics resources and accelerate the development of logistics systems. Tobacco companies have actively constructed smart cigarette delivery lockers to meet diverse customer needs and improve distribution efficiency. Smart cigarette delivery lockers are strategically placed in secure locations such as township governments, village service centers, and post offices. To complete self-service pickup, remote customers can confirm their identities through facial recognition and SMS verification codes. Figure 5 shows an example of a smart cigarette delivery locker.

The introduction of smart cigarette delivery lockers has dramatically improved the efficiency of cigarette distribution. These lockers enable retail customers to independently complete pickup and expand distribution service methods. It effectively prevents the occurrence of irregular business practices caused by inadequate Management of entrusted pickup points and pickup households, provides effective data support for standardized Management, and opens up the last mile of digital closed-loop management.

When selecting the location of the smart cigarette delivery locker, if some customers have to travel farther than other customers, it is considered an unfair service of the smart cigarette delivery locker. This paper's Goal is to ensure fairness in the distances between customers and smart cigarette delivery lockers.

Figure 5. A smart cigarette delivery locker.

5.2 Data Of The Case Study

This case study selects the districts and villages in Xiaopu Town, Jiangyong County, Hunan Province. Using the web service API provided by the Baidu Map open API, the longitude and latitude information of the villages in Xiaopu Town is crawled using the Python toolkit. We obtained 24 villages. Figure 6 shows the distribution of the cigarette demand in the villages of Xiaopu Town. The longitude and latitude of all the customers are shown in Table 10.

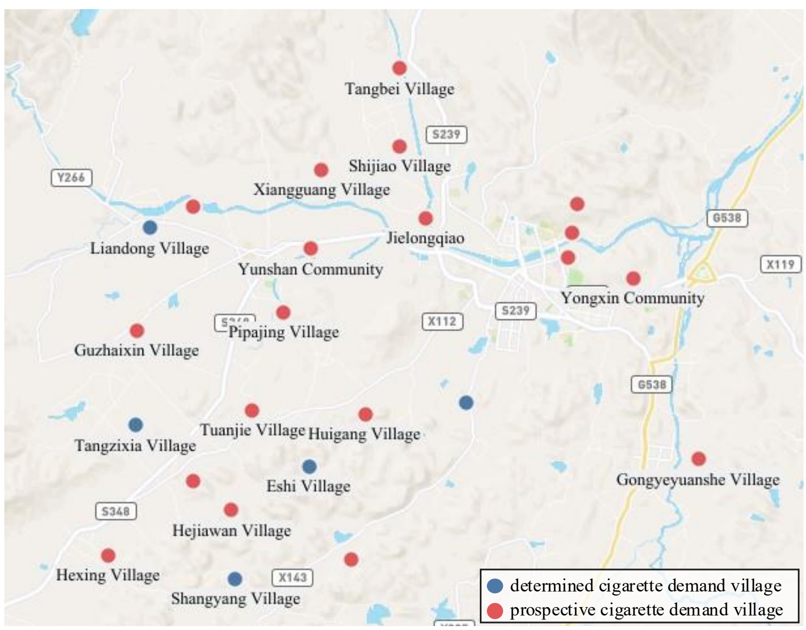
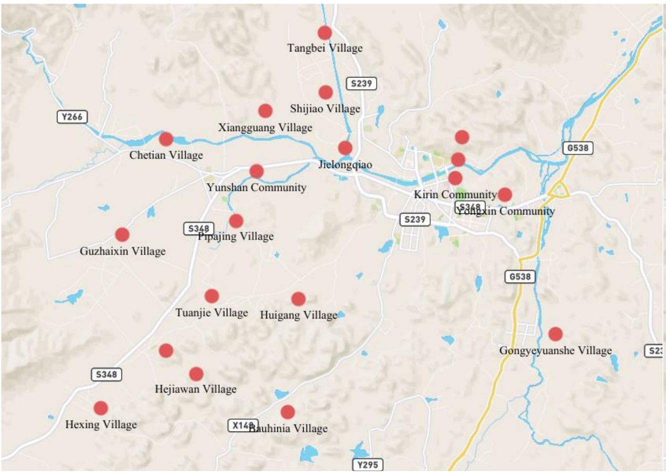


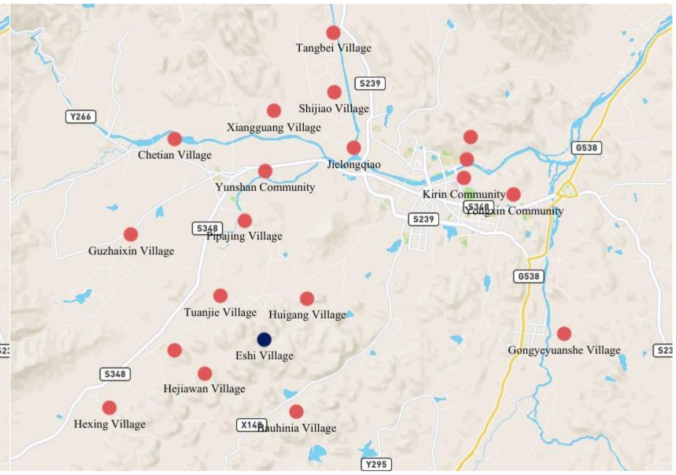
Figure 6. Distribution of cigarette demand in villages and smart candidate cigarette delivery lockers of Xiaopu Town.

This study considers 19 determined districts and villages as candidates for smart cigarette delivery locker locations and cigarette demand points, with an additional five villages designated as prospective cigarette demand points. Eight smart cigarette delivery lockers were selected from these candidates to serve other villages with high cigarette demand. Among them, the demand points serviced by each smart cigarette delivery locker do not exceed three cigarette demand points, which means $p = 3$.

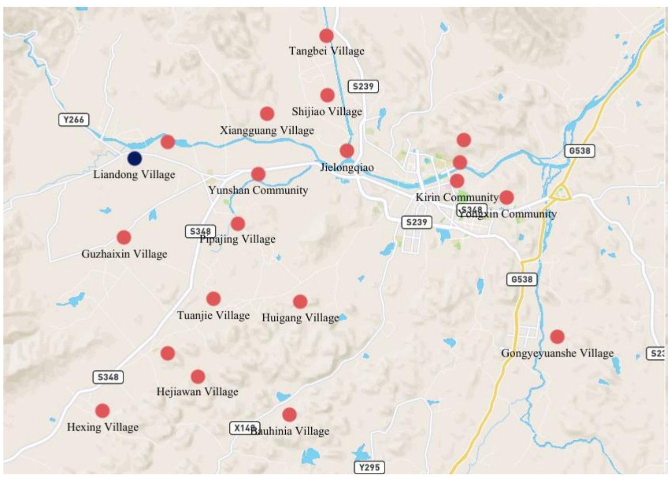
There are five prospective villages that could be used as cigarette demand points. As shown in Figure 6, red circles represent the determined cigarette demand villages, and the blue circles represent the prospective cigarette demand villages. As shown in Figure 7, there are 32 scenarios.



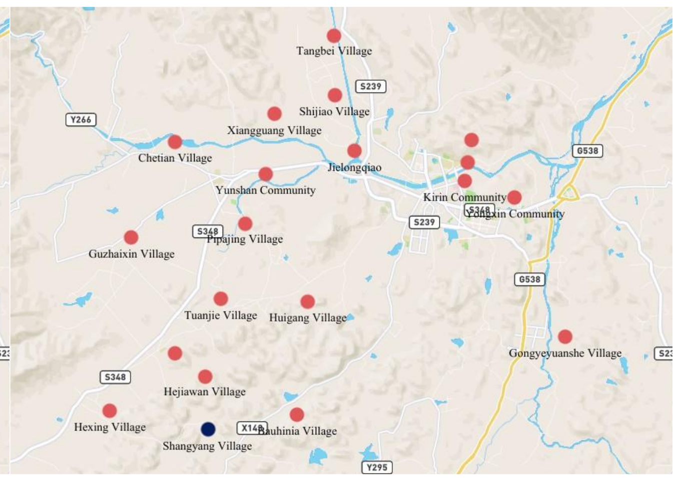
(a) Scenario 1



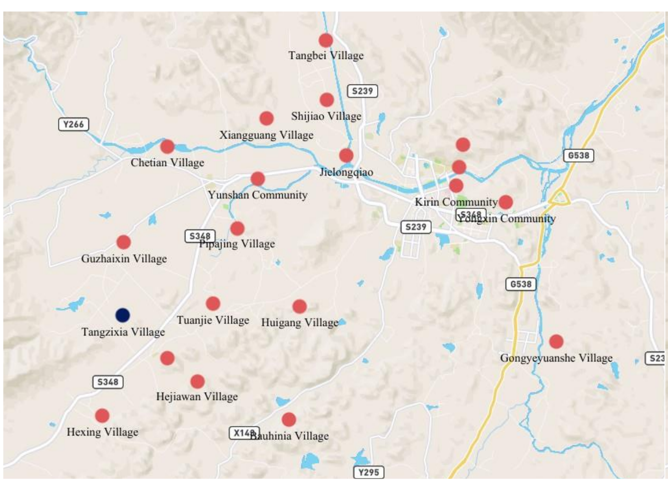
(b) Scenario 2



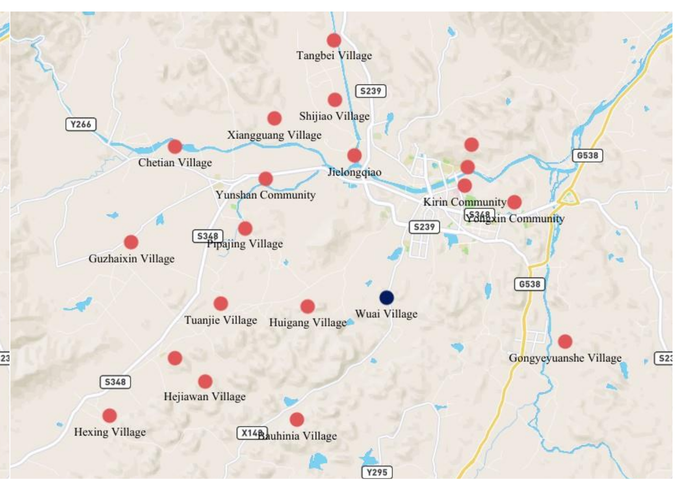
(c) Scenario 3



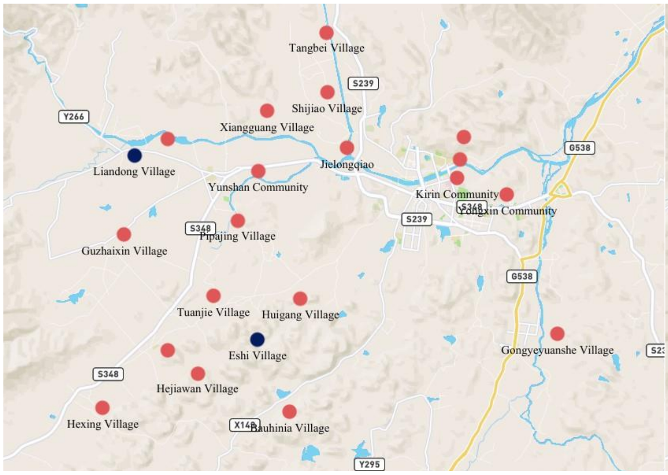
(d) Scenario 4



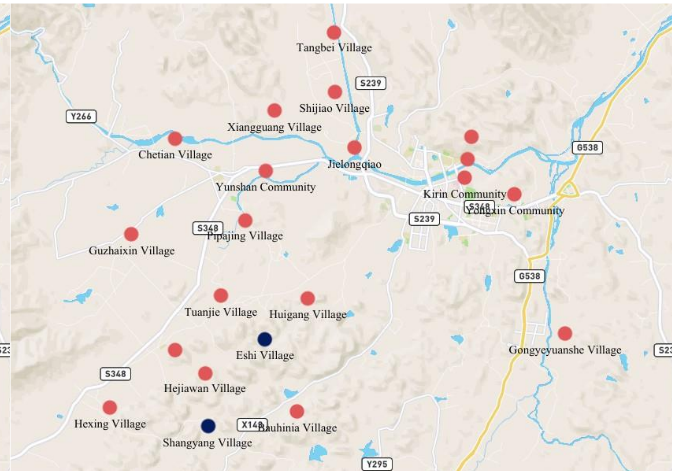
(e) Scenario 5



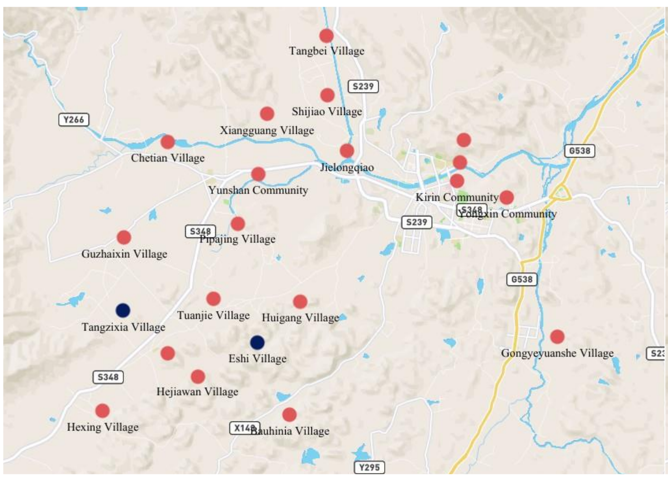
(f) Scenario 6



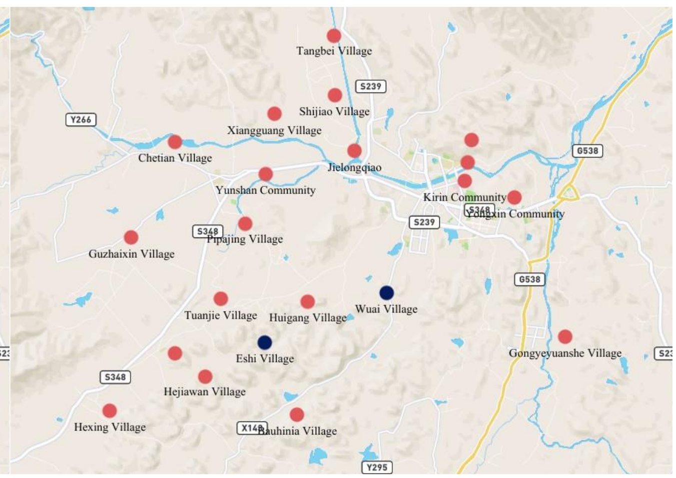
(g) Scenario 7



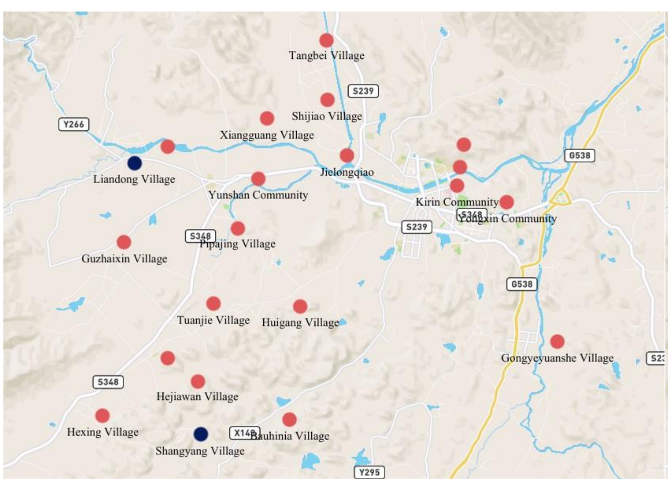
(h) Scenario 8



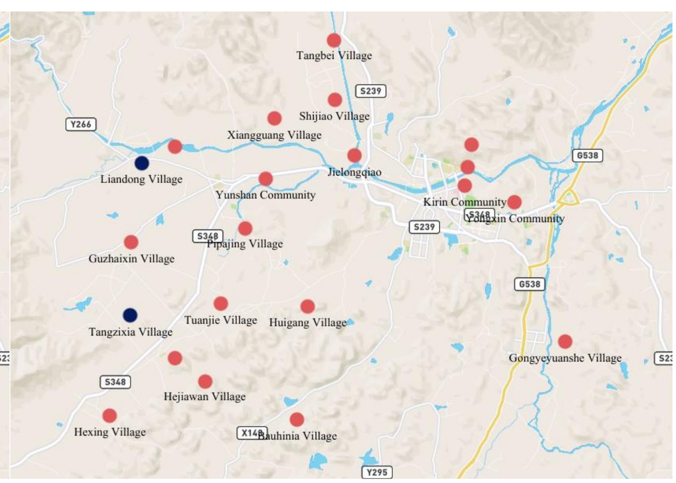
(i) Scenario 9



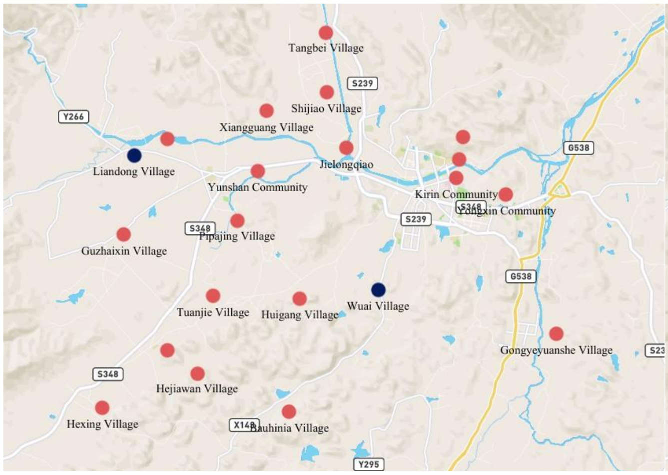
(j) Scenario 10



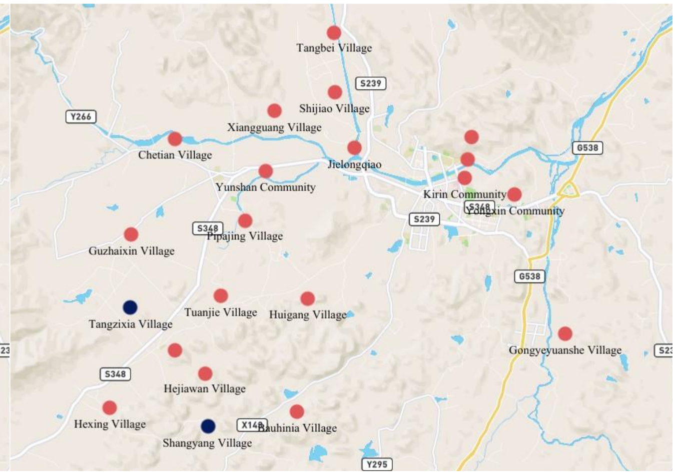
(k) Scenario 11



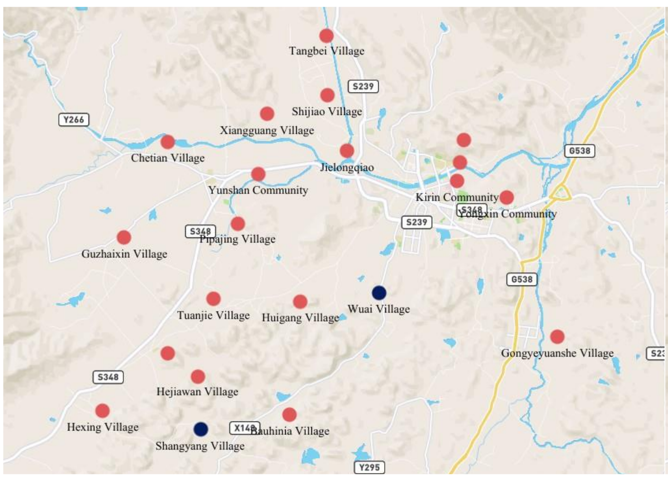
(l) Scenario 12



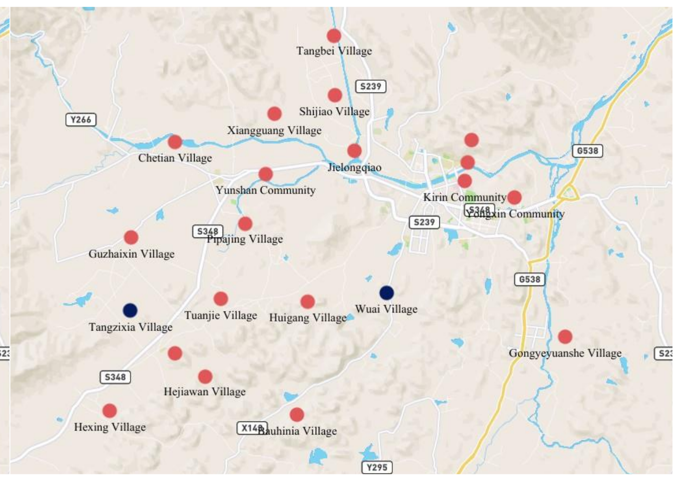
(m) Scenario 13



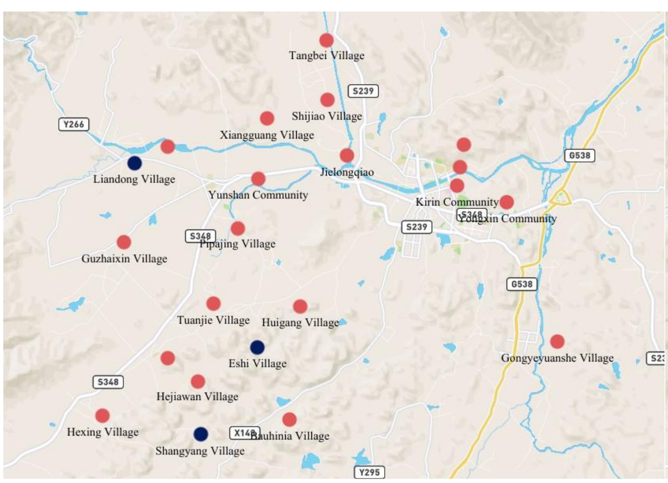
(n) Scenario 14



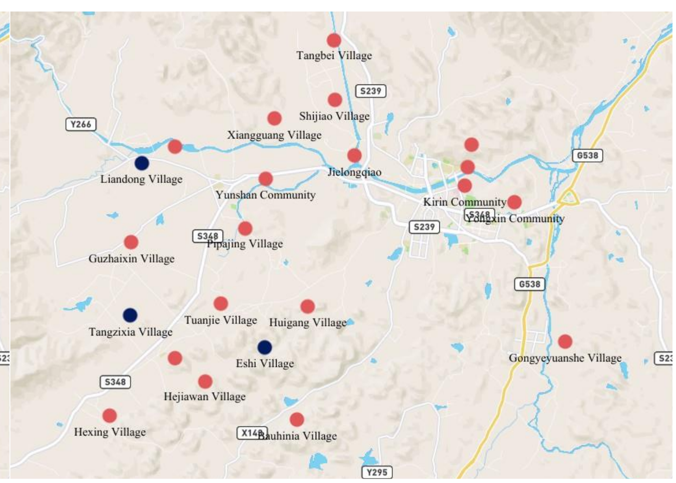
(o) Scenario 15



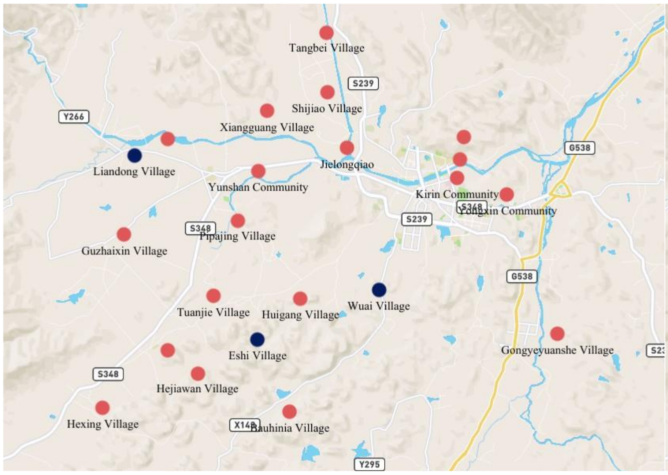
(p) Scenario 16



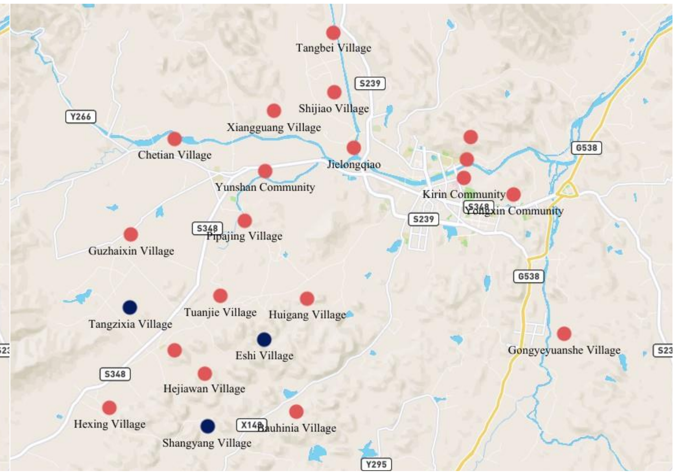
(q) Scenario 17



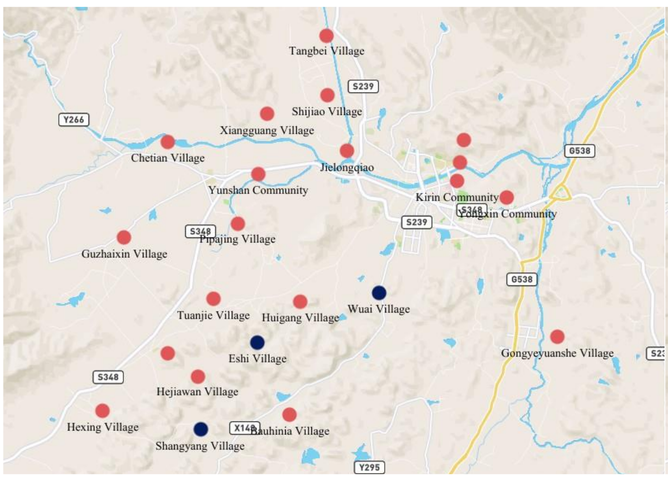
(r) Scenario 18



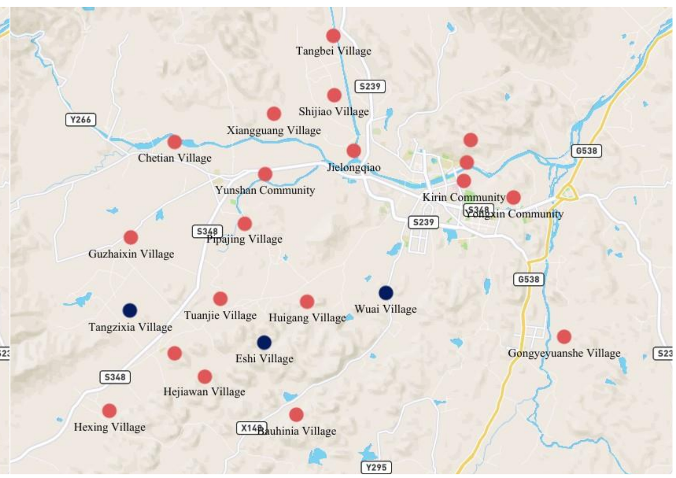
(s) Scenario 19



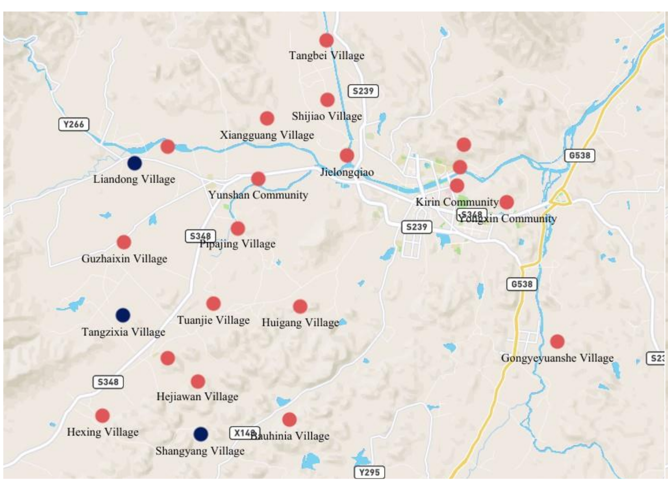
(t) Scenario 20



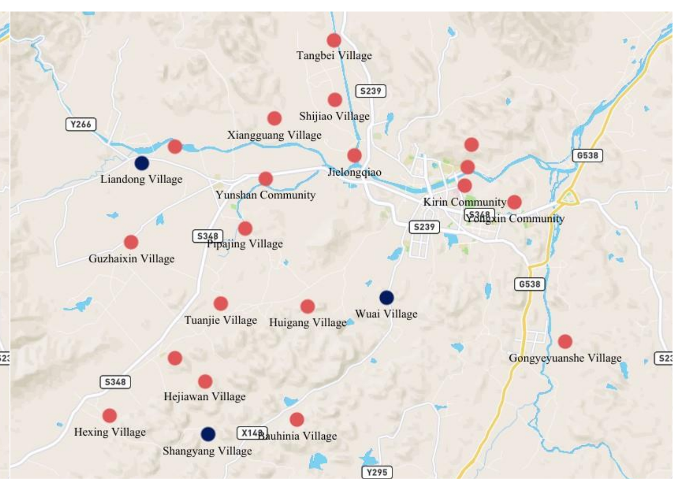
(u) Scenario 21



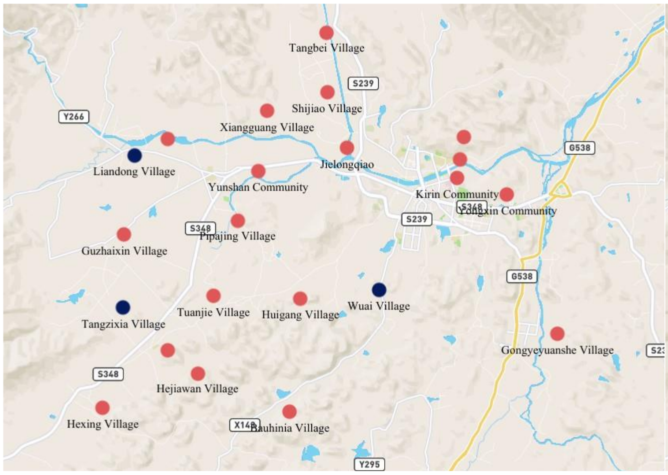
(v) Scenario 22



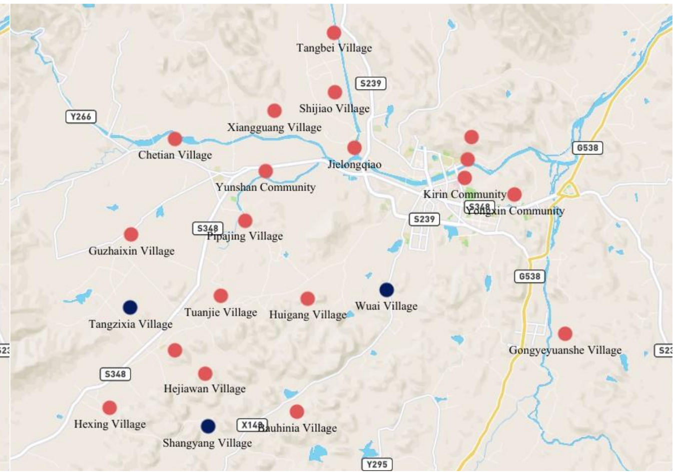
(w) Scenario 23



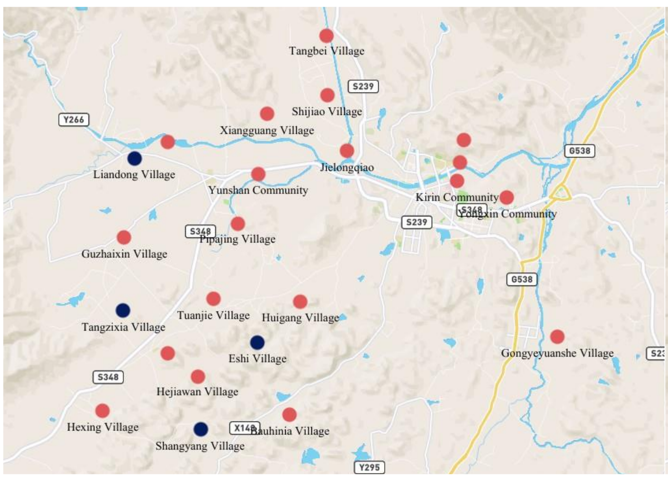
(x) Scenario 24



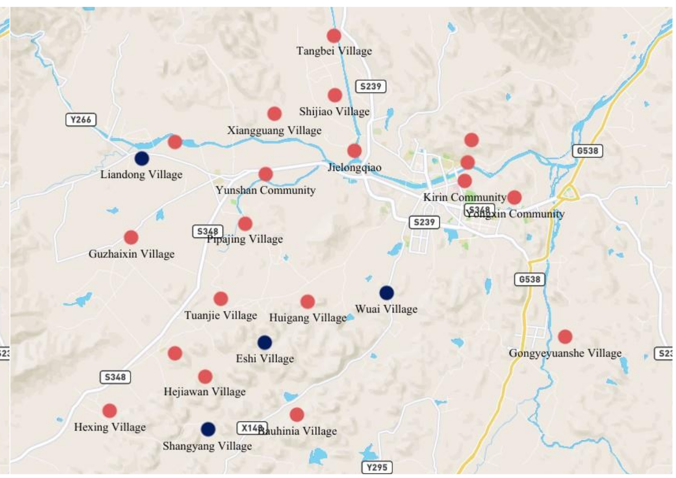
(y) Scenario 25



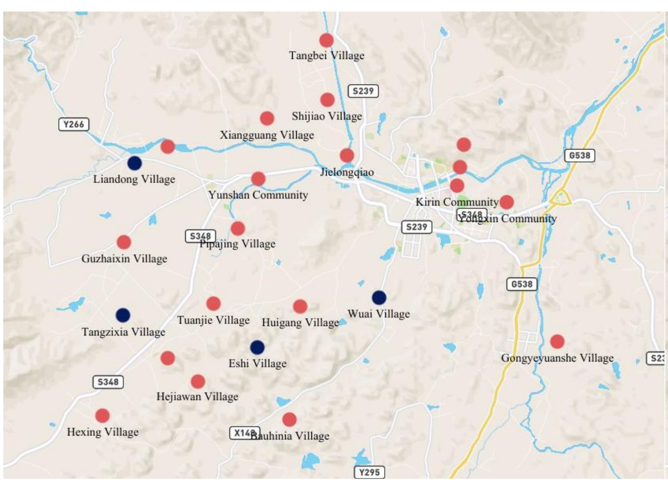
(z) Scenario 26



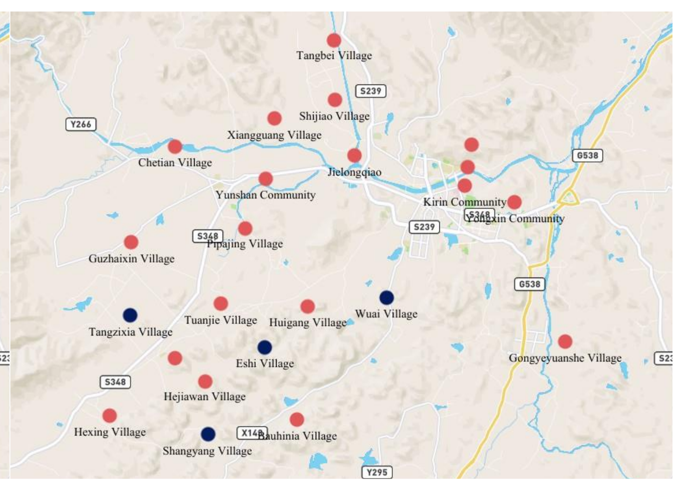
(aa) Scenario 27



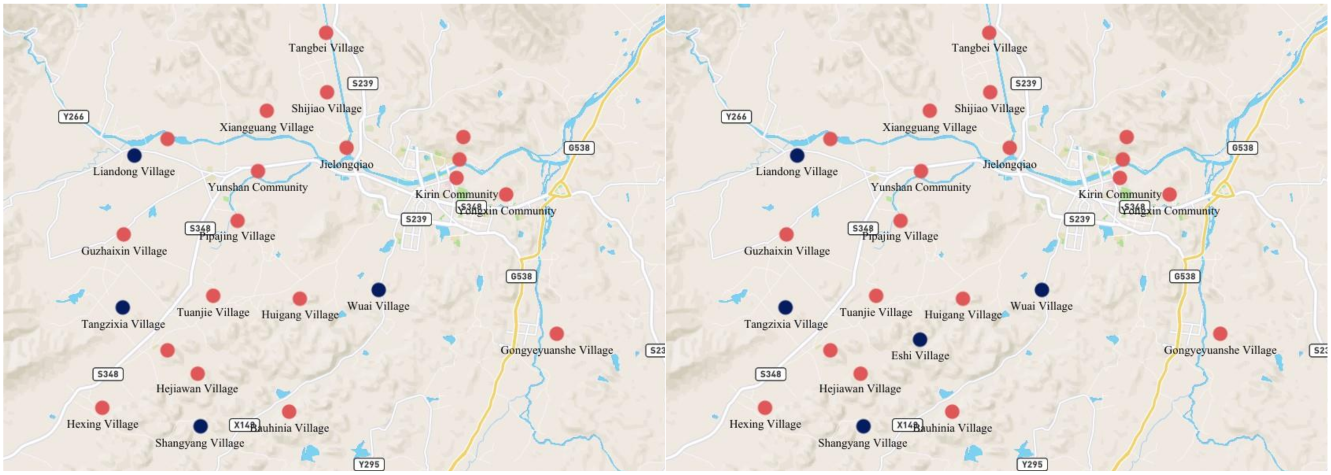
(ab) Scenario 28



(ac) Scenario 29



(ad) Scenario 30



(ae) Scenario 31 (af) Scenario 32

Figure 7. The distribution of determined and prospective customers for all scenarios.

5.3 A Scenario

In this section, we assume that there are no prospective customers and consider scenario one as the case based on the data of Xiaopu town. This section analyses the result of the model with no prospective customers. The computational result is shown in Figure 8 and Table 5.

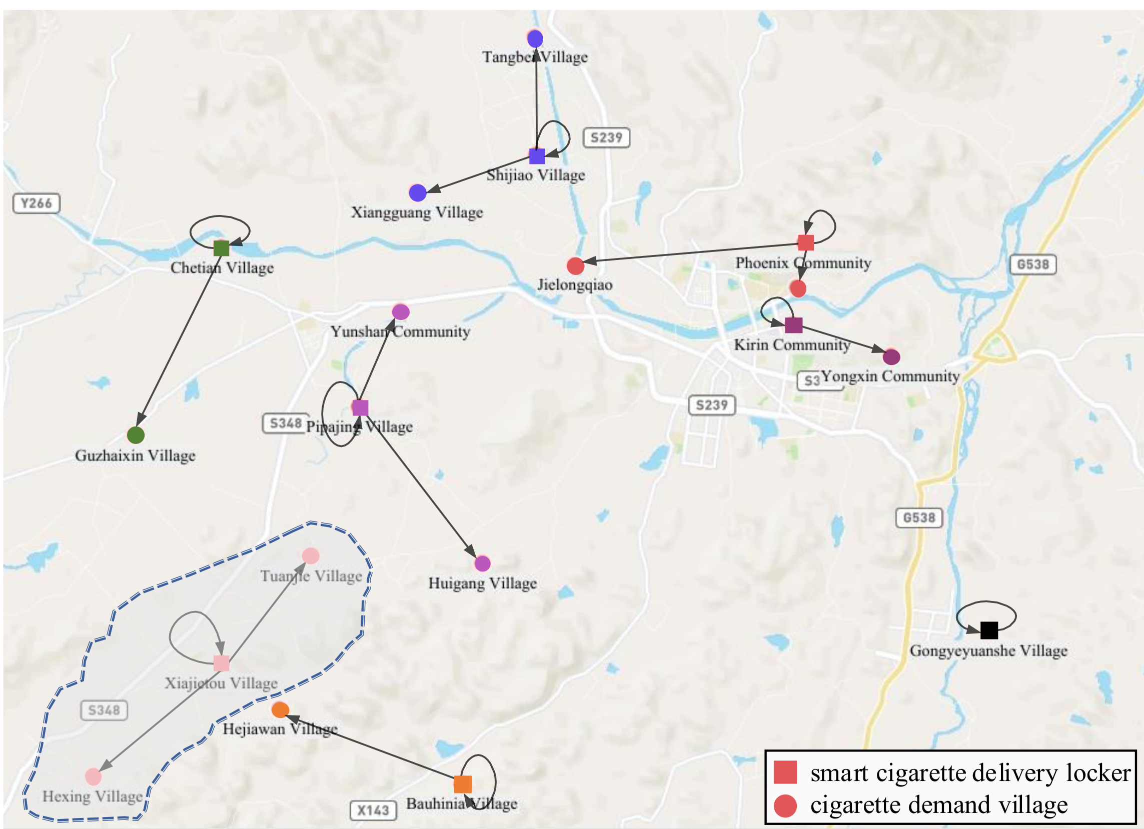


Figure 8. The distribution solution for the smart lockers is based on Scenario 1.

From Table 5 and Figure 8, we set smart cigarette delivery lockers in Phoenix Community, Kirin Community, Pipajing Village, Xiajietou Village, Chetian Village, Shijiao Village, Gongyeyuanshe Village, and Bauhinia Village. The maximum average distance from cigarette demand villages to each smart cigarette delivery locker is 1511.516, and the smart cigarette delivery locker location is Xiajietou Village, which services Xiajietou Village, Tuanjie Village, and Hexing Village.

Table 5. The distribution results of the smart lockers based on Scenario 1

Smart Cigarette Delivery Locker	Cigarette Demand Village
Phoenix Community	Sifangjing Community, Phoenix Community, JieLongqiao
Kirin Community	Kirin Community, Yongxin Community
Pipajing Village	Yunshan Community, Huigang Village, Pipajing Village
Xiajietou Village	Xiajietou Village, Tuanjie Village, Hexing Village
Chetian Village	Chetian Village, Guzhaixin Village
Shijiao Village	Xiangguang Village, Shijiao Village, Tangbei Village
Gongyeyuanshe Village	Gongyeyuanshe Village
Bauhinia Village	Hejiawan Village, Bauhinia Village
Optimal Solution	1511.516

5.4 Comparison Between Dinkelbach's Algorithm And Mathematical Modeling Method

This subsection compares the solutions obtained by different methods in terms of solution quality and computational time for different scenarios. In the above cases, this paper uses Dinkelbach's Algorithm (DA) and Reformulation-Linearization (RL) method to obtain the optimal solution. Table 6 shows the number of variables, constraints, objective values, and CPUs (computational time in seconds).

Table 6. Comparison of DA and RL methods for all scenarios.

Scenario	Algorithm	Variables		Constraints	Objective value	CPUs
		Discrete	Continuous			
1	DA	380	0	400	1511.516	1.824
	RL	380	380	1882	1511.516	2.328
2	DA	399	0	419	1595.774	2.626
	RL	399	399	1901	1595.774	1.310
3	DA	399	0	419	1511.515	1.803
	RL	399	399	1901	1511.515	2.314
4	DA	399	0	419	1511.515	3.020
	RL	399	399	1901	1511.515	3.041
5	DA	399	0	419	1595.774	3.495
	RL	399	399	1901	1595.774	1.820
6	DA	399	0	419	1687.919	1.967
	RL	399	399	1901	1687.919	3.331
7	DA	418	0	439	1595.774	2.503
	RL	418	418	2073	1595.774	2.182
8	DA	418	0	439	1595.774	2.982
	RL	418	418	2073	1595.774	2.624
9	DA	418	0	439	1613.393	2.347
	RL	418	418	2073	1613.393	1.640
10	DA	418	0	439	1687.919	2.332
	RL	418	418	2073	1687.919	1.672
11	DA	418	0	439	1511.515	1.957
	RL	418	418	2073	1511.515	3.961
12	DA	418	0	439	1595.774	2.712
	RL	418	418	2073	1595.774	3.327
13	DA	418	0	439	1687.919	3.281

Scenario	Algorithm	Variables		Constraints	Objective value	CPUs
		Discrete	Continuous			
	RL	418	418	2073	1687.919	3.004
14	DA	418	0	439	1595.774	2.826
	RL	418	418	2073	1595.774	2.186
15	DA	418	0	439	1687.919	2.579
	RL	418	418	2073	1687.919	2.218
16	DA	418	0	439	1702.281	2.631
	RL	418	418	2073	1702.281	2.209
17	DA	437	0	459	1595.774	3.683
	RL	437	437	2169	1595.774	2.621
18	DA	437	0	459	1613.393	3.414
	RL	437	437	2169	1613.393	2.233
19	DA	437	0	459	1687.919	1.646
	RL	437	437	2169	1687.919	3.051
20	DA	437	0	459	1702.281	1.959
	RL	437	437	2169	1702.281	2.342
21	DA	437	0	459	1717.813	1.959
	RL	437	437	2169	1717.813	3.331
22	DA	437	0	459	1702.281	2.963
	RL	437	437	2169	1702.281	4.256
23	DA	437	0	459	1595.774	3.204
	RL	437	437	2169	1595.774	2.151
24	DA	437	0	459	1687.919	2.836
	RL	437	437	2169	1687.919	2.511
25	DA	437	0	459	1710.927	2.808
	RL	437	437	2169	1710.927	2.394
26	DA	437	0	459	1702.281	2.808
	RL	437	437	2169	1702.281	2.292
27	DA	456	0	479	1919.174	3.257
	RL	456	456	2265	1919.174	4.889
28	DA	456	0	479	1919.174	2.249
	RL	456	456	2265	1919.174	2.548
29	DA	456	0	479	1919.174	2.062
	RL	456	456	2265	1919.174	3.519
30	DA	456	0	479	1919.174	2.033
	RL	456	456	2265	1919.174	2.462
31	DA	456	0	479	1919.174	2.371
	RL	456	456	2265	1919.174	2.085
32	DA	475	0	499	1919.174	1.143
	RL	457	475	2361	1919.174	4.431

From the above table, we can find that DA and RL could obtain the same optimal solutions for all the different scenarios. Both DA and RL could obtain the optimal solutions. Figure 9 shows the comparison of computational time for DA and RL.

The computational results of Figure 9 indicate that DA's is more robust than the RL method in terms of computational time. In many mathematical models, we need to set a "Big M" (Rubin, 2011). Different M values will incur different rounding errors and running times. In this paper, we also need the "Big M" to solve RL. Determining an appropriate value of M is also time-consuming. In this paper, we used the same $M = 50000$ for all the cases.

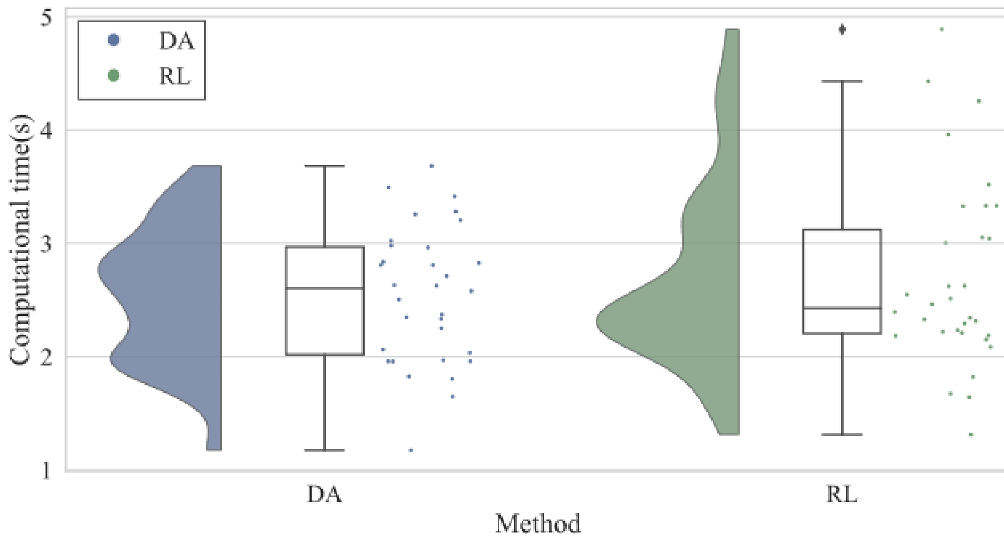


Figure 9. Comparison of computational time for DA and RL.

5.5 Comparison With Different Objective Functions

Inspired by this study of Huang *et al*. (2012a), in this section, we compare two different goals for the above-studied problem in terms of efficiency and fairness.

This paper's Goal is to minimize the maximum average distance between customers (cigarette demand villages) and the corresponding smart cigarette delivery locker. The objective function is Min $\left(\max_{i\in P}\left\{\frac{\sum_{j\in C_s}(c_{ij}*y_{ij})}{\sum_{j\in C_s}y_{ij}}\right\}\right)$. The other Goal considered by the previous studies is to minimize the maximum total distance between cigarette demand villages and the corresponding smart cigarette delivery lockers. The objective function is Min $\left(\max_{i\in P}\left\{\sum_{j\in C_s}c_{ij}*y_{ij}\right\}\right)$ based on efficiency. Another goal considered by previous studies is to minimize the maximum distance between each cigarette demand village and the corresponding smart cigarette delivery locker. The objective function is Min $\left(\max_{i\in P}\left\{c_{ij}*y_{ij}\right\}\right)$ based on fairness.

To compare these goals, in this section, we define Goal (Z_f) as the model and set Z_f as the metric. Let $Z_1 = \max_{i\in P}\left\{\frac{\sum_{j\in C_s}(c_{ij}*y_{ij})}{\sum_{j\in C_s}y_{ij}}\right\}$, $Z_2 = \max_{i\in P}\left\{\sum_{j\in C_s}c_{ij}*y_{ij}\right\}$ and $Z_3 = \max_{i\in P}\left\{c_{ij}*y_{ij}\right\}$. Note that Z_1 is the objective considered by this paper. The models are summarized as follows.

$$\text{Min} \quad Z_f \tag{51}$$

Subject to constraints (9)-(13).

Table 7. Comparison of the optimal solutions of different objective functions.

Optimal solution	Model			Total distance
	Z_1	Z_2	Z_3	
Z_1^*	**1511.515***	4534.695	3362.366	**23880.844**
Z_2^*	1627.538	**3810.277***	3255.238	23979.462
Z_3^*	2393.859	7181.815	**3009.267***	29554.176

*denotes the optimal solution for each row.

Table 7 compares three different models in terms of different objective functions related to smart cigarette delivery lockers and corresponding cigarette demand villages with the same constraints. In each row of Table 7, we optimize Z_1, Z_2 and Z_3, respectively. The red colored font denotes the optimal Z_f. By using the optimal Z_f^*, we calculate other $Z_{f'}$ in each row, where $f \neq f'$. From Table 7, when optimizing the Z_1, we obtain the minimum total distance. To compare each column of Table 7, we define Gap_f, which is shown as follows.

$$Gap_f = \frac{Z_f - Z_f^*}{Z_f^*} * 100\% \tag{52}$$

Table 8 shows the between Z_f and Z_f^*. From Table 8, we can find that there are two zeros in row one.

For Table 8, compared to the objective function in this paper, the maximum distance of Z_2 and Z_3 is increased by 19.01% and 11.73%, respectively.

Figure 10 shows the average distance between each pickup hub and customers for different goals. The blue markers denote the selected pickup hub. From Figure 10, we can find that the distribution of the first column is more compact. The red cube in Figure 10 represents the average distance between all the pickup hubs and customers. This paper aims to have the minimum average value between all the pickup hubs and customers. We can find that the red marker of Z_1^* is the minimum among these three different goals. Figure 11 shows the optimal solutions for Z_1,Z_2, and Z_3.

Table 8. The gap value between Z_f and Z_f^*.

Optimal solution	Gap_f			The gap in total distance
	$f = 1$	$f = 2$	$f = 3$	
Z_1^*	0	19.01%	11.73%	0
Z_2^*	7.68%	0	8.17%	0.41%
Z_3^*	58.37%	88.49%	0	23.76%

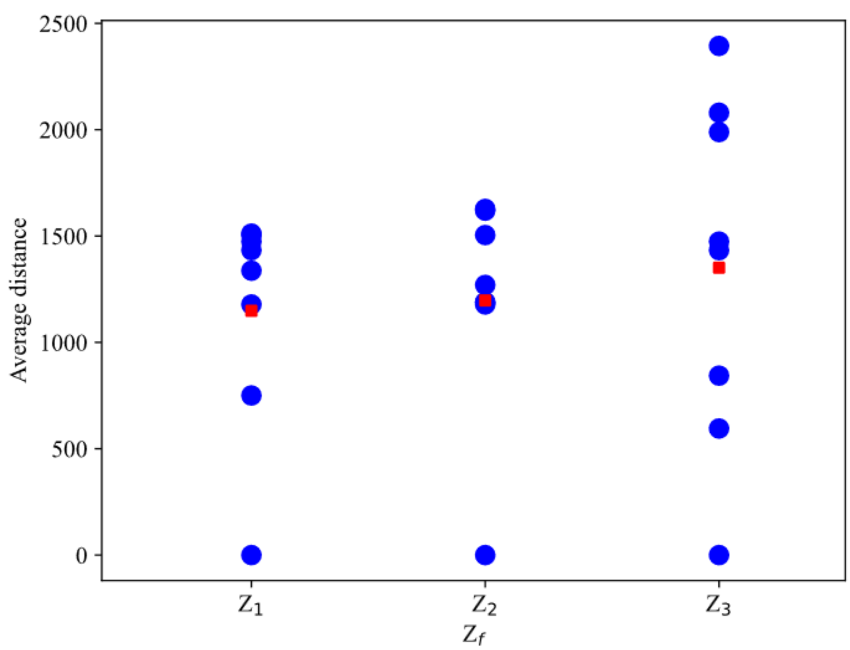
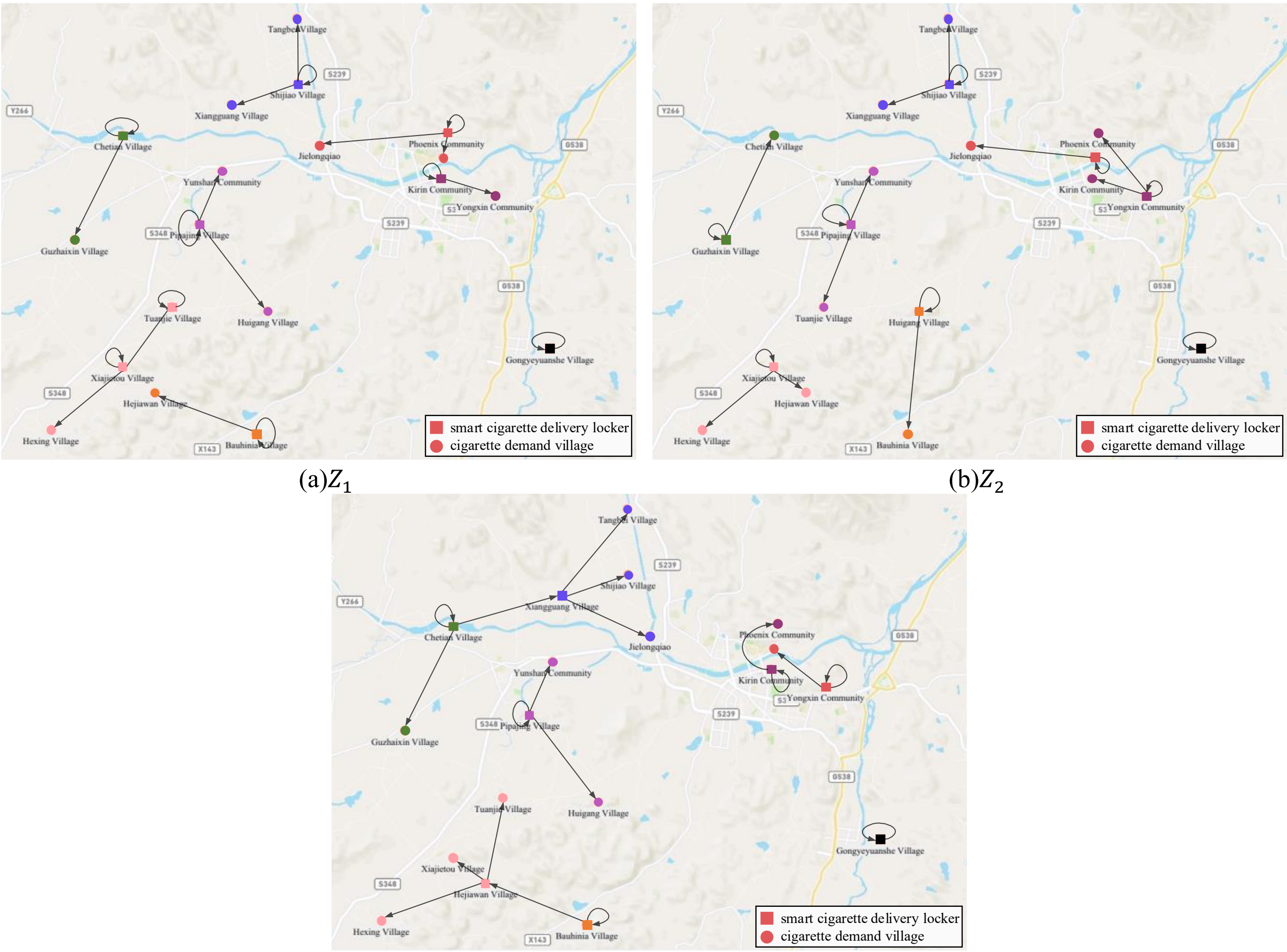


Figure 10. Average distance between each pickup hub and customers for different goals.



(a)Z_1 (b)Z_2

(c)Z_2

Figure 11. The distribution results correspond to the optimal solutions of Z_1,Z_2, and Z_3.

5.6 Comparison With the P-Median Model

To validate the effectiveness of our proposed model, we compared it with the classical p-median model. The p-median model is one of the most widely used approaches in facility location problems, focusing on minimizing the total cost between facilities and demand points. However, the p-median model typically cannot handle fairness constraints directly. Therefore, through comparative experiments, we demonstrate the advantages of our model in terms of fairness.

We conducted the comparative experiments using the scenario 1 dataset. The experimental evaluation metrics were set as follows:

Total Cost: The sum of distances between pickup hubs and demand points.

Fairness Index: The maximum value of the average of the distances between pickup hubs and demand points.

Table 9 compares our proposed model and the p-median model in terms of total cost and fairness index.

Table 9. The optimal solutions of this paper and the p-median model

Evaluation Metrics	This paper Model	p-median Model	Gap
Total Cost	23880.844	20203.383	18.20%
Fairness Index	1511.515	1912.525	26.53%

From Table 9, we observe the following:

(1) Regarding total cost, the p-median model slightly outperforms our proposed model because it is specifically designed to achieve cost minimization. The total distance is reduced by 18.20% compared to our model.

(2) Regarding fairness, our proposed model outperforms the p-median model, with the maximum average distance reduced by 26.53%.

5.7 Sensitive Analysis of λ

The key to the Dinkelbach algorithm is to iterate and update the value of λ. $F(\lambda)$ is continuous and strictly decreasing, and the sequence λ_k is monotone decreasing (See pages 305-306 of Lev (2006)). In this section, we analyze the different values of λ in terms of the computational time for the Dinkelbach algorithm. Since the algorithm requires an initial λ, its choice significantly affects the algorithm's convergence. In the following experiments, we set λ varying from 0 to 2000, using Scenario 1 for evaluation. Figure 12 shows the convergence analysis of Dinkelbach's algorithm with different λ. The gray line is the optimal λ.

From Figure 12, we observe that λ decreases over iterations, and the convergence rate accelerates as λ approaches the global optimum. Therefore, we can set the initial value of λ to improve the solution efficiency of Dinkelbach's algorithm. For example, we can analyze the data by the k-means clustering algorithm to obtain an initial solution and get a λ, which is close to the optimum.

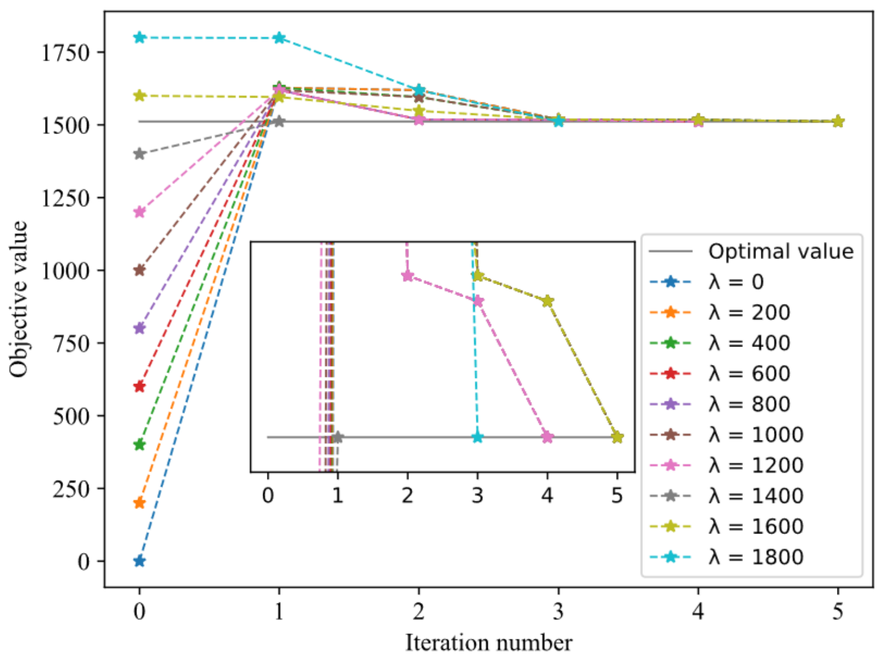


Figure 12. Convergence analysis of Dinkelbach's algorithm with different λ.

Figure 12 shows the computational time and maximum iterations for each λ. From Figure 12, we can find that λ closing to the optimal λ has less number of maximum iterations.

Figure 13 visually shows the effect of λ on operating efficiency. It shows the initial value of lambda as the horizontal axis (0-1000), the number of iterations (0-16 times), and the running time (0-10 seconds) as the vertical axis. The bar chart indicates the running time, and the line chart reflects the operational efficiency.

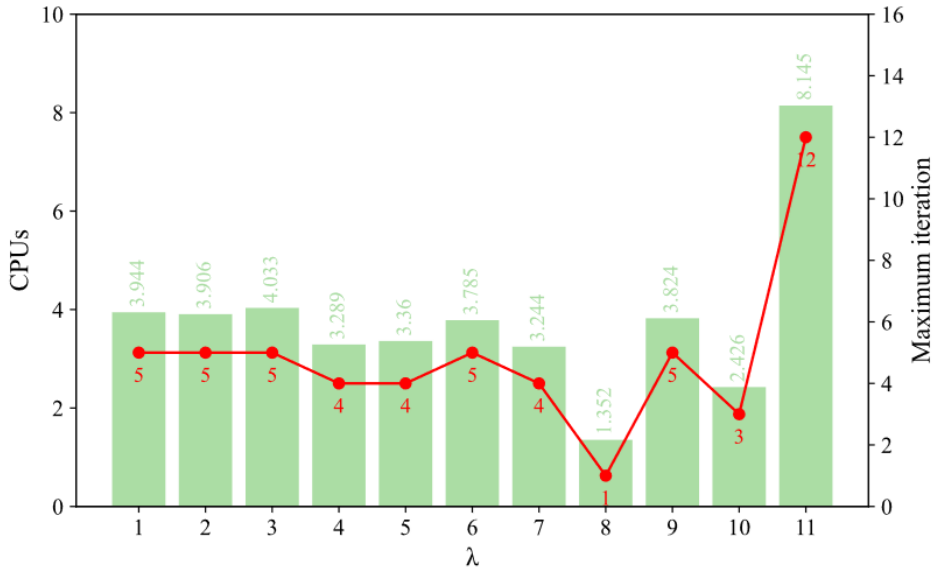


Figure 13. Computational time and maximum iterations for each λ.

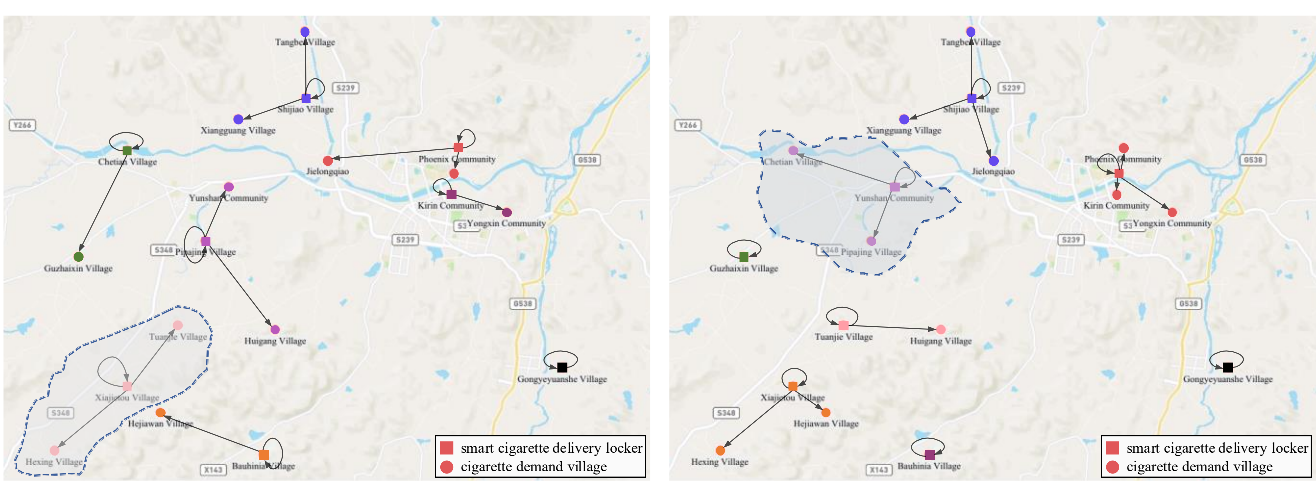
5.8 Analysis Of The Number Of Pickup Stations

To illustrate the impacts of different numbers of demand points each pickup station served, this section compares different numbers of pickup stations for computational results. As shown in Table 10, the optimal solution becomes better with the value of p increasing, which demonstrates the advantages of more pickup stations. However, the optimal solutions are the same when p equals 4, 5, 6, 7, and 8.

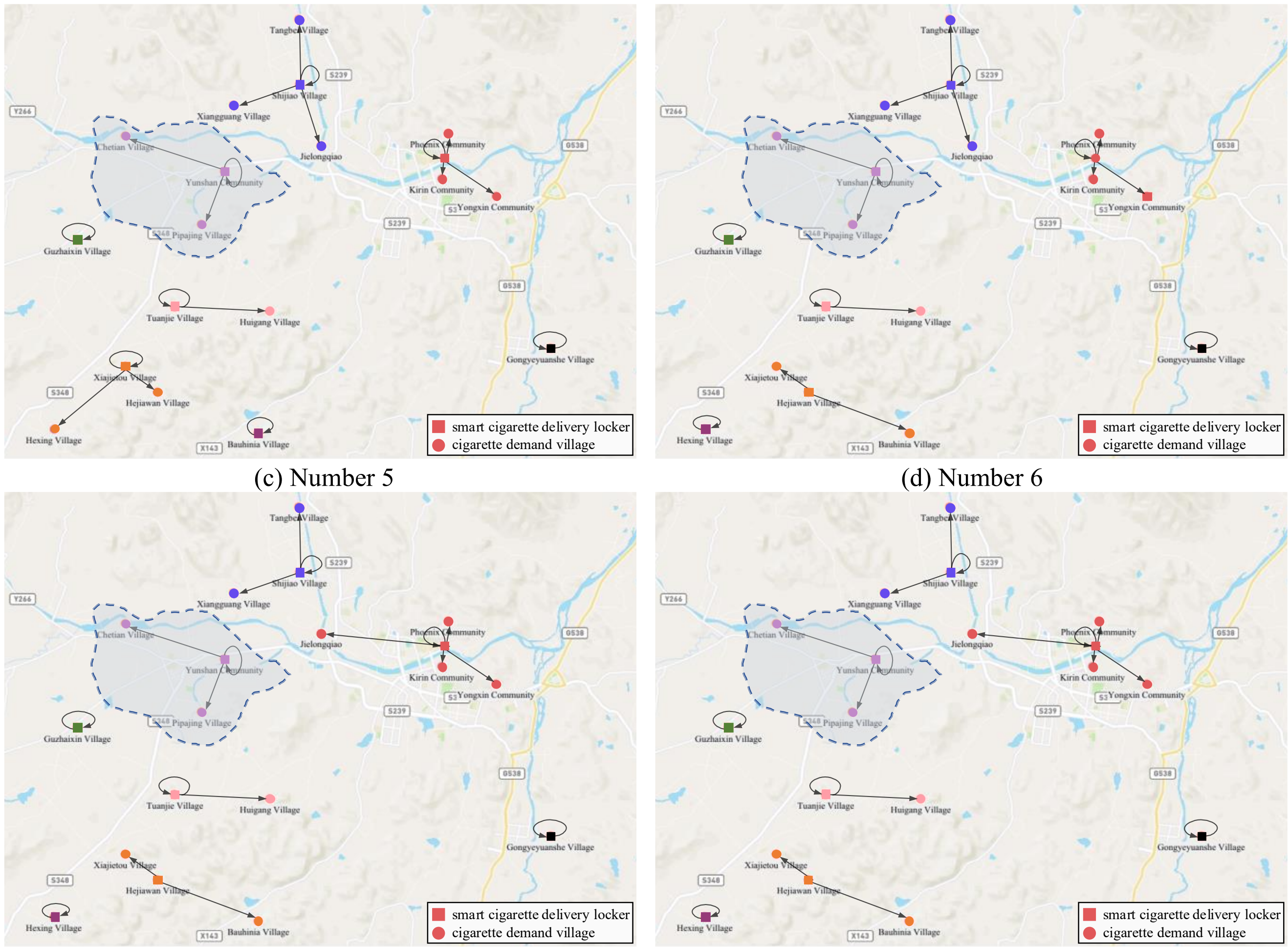
Table 10. Optimal solutions for different numbers of pickup stations.

Number of pickup stations (p)	3	4	5	6	7	8
Optimal solution	1511.515	1430.041	1430.041	1430.041	1430.041	1430.041

Figure 14 shows the distribution results of different numbers of pickup stations for scenario 1. The dotted polygon denotes the maximum average distance between customers and pickup hubs in Figure 14. From Figure 14, we can find that the maximum average distance between customers and pickup hubs is the same for p, equaling 4, 5, 6, 7, and 8.



(a) Number 3 (b) Number 4



(c) Number 5 (d) Number 6

(e) Number 7 (f) Number 8

Figure 14. The distribution results of the different numbers of pickup stations for scenario 1.

5.9 Optimal Solutions For All Scenarios

This section compares the optimal solutions for all scenarios. Figure 15 shows the optimal solution for each scenario. Due to the random prospective cigarette demand villages, this paper chooses the maximum of the optimal solution of all scenarios instead of one scenario. From Figure 15, the worst solution is 1919.174 with corresponding scenarios 27, 28, 29, 30, 31, and 32.

Furthermore, to illustrate the details of different scenarios, this paper calculates the average distance of each pickup station for all Scenarios. As shown in Figure 16, with the prospective cigarette demand villages changing, the average distance of each pickup station is changing.

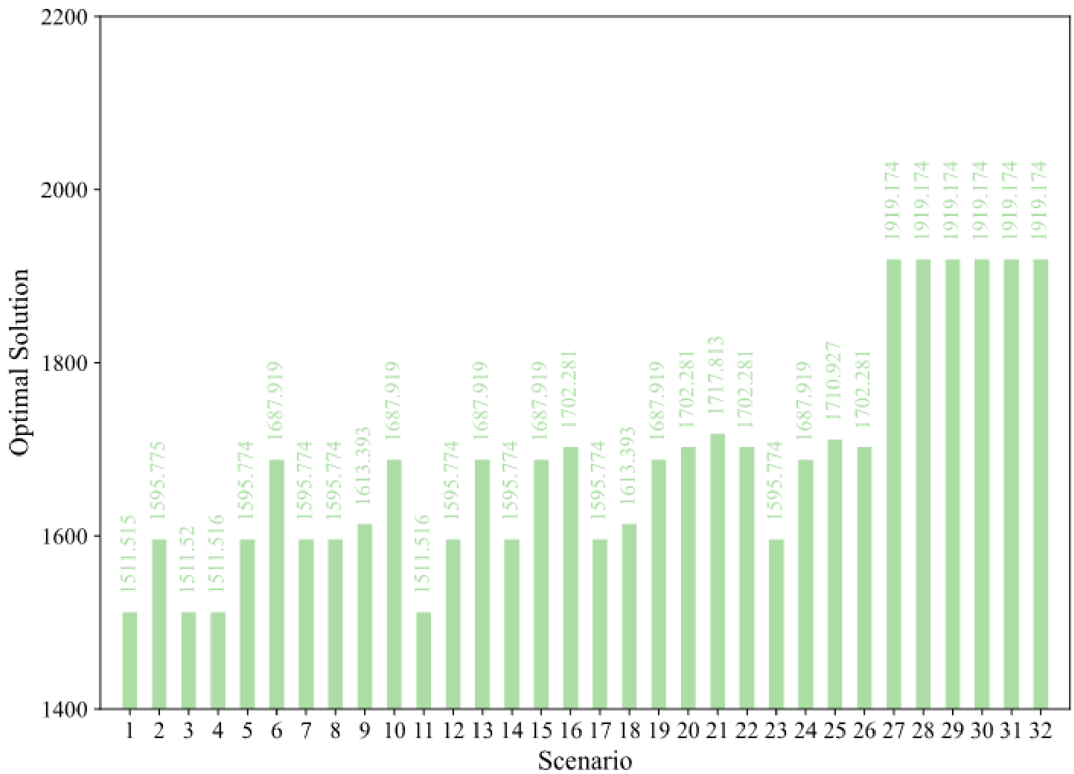


Figure 15. The optimal solution for each scenario.

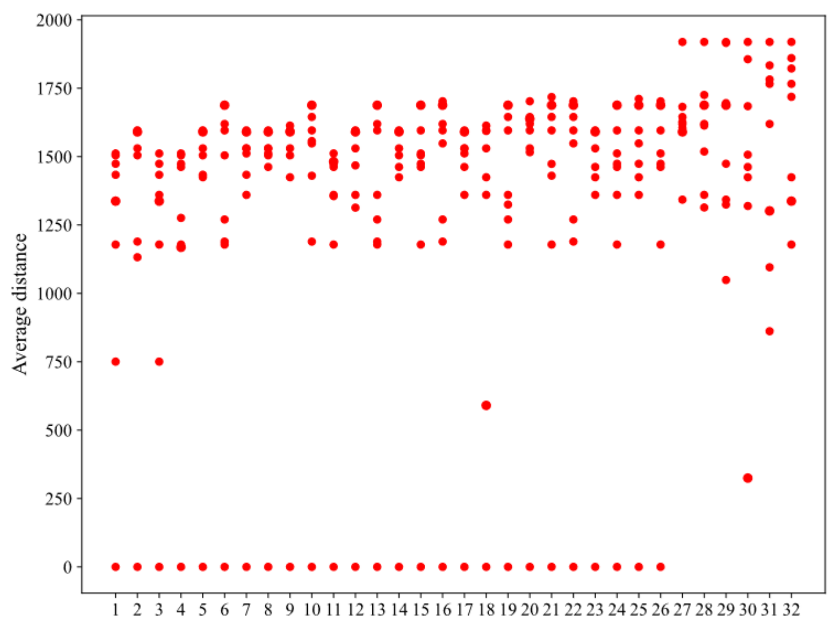


Figure 16. The average distance of each pickup station for all scenarios.

5.10 Management Insights

This paper investigates the pickup hub location problem considering prospective customers. Two scenario-based solution methods are proposed for the studied problem. Even though providing doorstep service will increase convenience for customers, it will increase the transportation cost for logistics companies. If the logistics companies increase the delivery fee, they will experience customer loss. The rural areas have lower average incomes, and the local government wants to promote the rural economy. To provide doorstep delivery service in rural areas, the local government will provide subsidies to the logistics companies that provide doorstep delivery service at the specified pickup hubs in rural areas. To ensure equality in accessing pickup hubs, the local government could adopt the model proposed by this paper. This paper could help the local government determine the pickup hub's location.

6. CONCLUSIONS

In rural areas of China, most logistics companies do not provide doorstep delivery services, with the exception of China Postal Express & Logistics. Without using this service, customers need to go to the nearest pickup hub to get their orders when they buy goods online. Now, even the nearest pickup hub is very far away from the customer's home in rural areas. To improve delivery efficiency, postal services could install pickup hubs near villages in rural areas. Popular local facilities, such as florists, coffee shops, clothing boutiques, gas stations, plumbers, and hair salons, can be considered as the candidate pickup hubs. With the limited budget, the local government could set up pickup hubs for all the villages in rural areas. To ensure the equality of accessing the pickup hubs for customers to pick up their orders, at most three Villages must share a pickup hub. Under the above background and assumptions, this paper studied the pickup hub location problem with prospective customers considering equality and efficiency. A nonlinear fractional programming model is developed to formulate the studied problem, and a linearization model is also introduced to ensure that mixed integer solvers, such as CPLEX, can solve the studied problem. A scenario-based Dinkelbach's algorithm and scenario-based mathematical formulation methods are proposed to solve the problem being studied. A case study is conducted to illustrate the proposed model and verify the efficiency of the proposed solution methods.

Through case analysis, this paper's location selection scheme can significantly reduce the service gaps between different villages. The case study shows that the maximum distance for this paper's fairness goal is minimal compared to other objectives. Therefore, the pickup hub location optimization based on distance equity in this paper not only improves the accessibility of services in rural areas but also promotes the reasonable allocation of social resources. This paper's conclusion highlights the importance of equity in policy-making and service delivery, especially when it comes to underserving vulnerable groups. This paper ensures that every village has equal access to essential services, which effectively reduces regional disparities and improves overall social well-being. In addition, the study provides a practical reference for local governments and related agencies to help them achieve a more balanced service layout in the context of limited resources.

Future research could explore the following directions: (1) We can design meta-heuristic methods for solving very large-scale problems. (2) The customer's demand could be considered when formulating a model with demand uncertainty. (3) This paper could also be extended to the last-mile distribution network design.

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APPENDIX A: Geographic location of pickup hubs

Table 11. Geographic location information of the determined and prospective customers.

	Pickup hub	Longitude	Latitude
Determined location	Sifangjing Community	111.352199	25.27868
	Phoenix Community	111.353295	25.284436
	Kirin Community	111.351517	25.273855
	Yongxin Community	111.36571	25.26968
	Yunshan Community	111.294742	25.275575
	Huigang Village	111.306678	25.24259
	Pipajing	111.882707	26.466784
	Xiangguang Village	111.297086	25.291244
	Xiajietou Village	111.268837	25.229324
	Hejiawan Village	111.27729	25.223439
	Chetian Village	111.268834	25.283844
	Guzhaixin Village	111.256331	25.259256
	Shijiao Village	111.314336	25.295974
	Tangbei Village	111.314213	25.311471
	Gongyeyuanshe Village	111.380074	25.233668
	Tuanjie Village	111.281848	25.243394
	Jielongqiao	111.319993	25.281673
	Bauhinia Village	111.303632	25.213593
	Hexing Village	111.250185	25.214534
Prospective location	Eshi Village	111.294376	25.232156
	Wuai Village	111.329015	25.244968
	Shangyang Village	111.278135	25.209797
	Liandong Village	111.259235	25.279671
	Tangzixia Village	111.256073	25.240392